BASIC FACTS OF GROUP THEORY.
HOMEWORK 1.

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1. Groups of transformations and abstract groups.

The notion of an abstract group came from more concrete notion of group of transformations. Suppose, we have some set X and a collection G of **transformations** of X. Here by transformation we understand a map $f: X \to X$.

Such a collection G is called a **group** if the following is true:

a) If g_1, g_2 are two transformations from G, then their composition $g_1 \circ g_2$ also belongs to G. Recall that $g_1 \circ g_2$ denotes the consecutive application of g_2 and g_1.

b) Every transformation $g \in G$ is invertible and the inverse transformation g^{-1} also belongs to G.

A typical example of a group of transformation is the set of transformations which preserve one or several properties of elements of X. For example, if X is a metric space with a distance function $d(x, y)$, then the collection of all invertible **isometries**, i.e. transformations g such that $d(g \cdot x, g \cdot y) = d(x, y)$, form a group.

For us the most important example is the group $GL(V)$ of all invertible linear transformations of a vector space V. By definition, it consists of invertible transformations $A: V \to V$ such that $A(v_1 + v_2) = Av_1 + Av_2$ and $A(\lambda v) - \lambda Av$.

If we choose a basis in V, the group $GL(V)$ can be identified with the group $GL(n, K)$ of invertible matrices with elements from the basic field K (which will be usually $\mathbb{R}$ or $\mathbb{C}$). Subgroups of $GL(V)$ or $GL(n, K)$ are called **linear groups**.

An **abstract group** is what remains from a group of transformations if we forget about transformations and consider G as a set endowed with the multiplication law, satisfying the associativity axiom: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.

Note that in case of groups of transformations we did not require the associativity since it is an intrinsic property of the operation of composition.

It is not difficult to show that every abstract group can be realized as a group of transformations.

It turns out that many quite different groups of transformations can be isomorphic as abstract groups. Here is an example:

The same abstract group can be considered as

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- The group of isometries of an equilateral triangle.
- The group $GL(2,\, \mathbb{F}_2)$ where $\mathbb{F}_2$ is a field with two elements.
- The group of Möbius transformations generated by $z \to 1 - z$ and $z \to \frac{1}{z}$.
- The group of projective transformation of the projective line over $\mathbb{F}_2$.
- The group S_3 of all permutations of three objects.

2. Groups generated by reflections and Coxeter groups.

A very interesting and important class of linear groups is the groups generated by reflections. Recall that a linear operator s in Euclidean space $\mathbb{R}^n$ is called **reflection** if it preserves the inner product and all points of a hyperplane $M \subset \mathbb{R}^n$ If the equation of M is $(v,\, a) = p$ for some non-zero vector $a \in \mathbb{R}^n$ and some $p \in \mathbb{R}$, then

$$s(v) = v + 2\,\frac{p - (v,\, a)}{(a,\, a)}\, a. \tag{1}$$

It is clear that $s^2 = \mathrm{Id}$, so all groups generated by one reflection are isomorphic to the abstract group $C_2 = \mathbb{Z}/2\mathbb{Z}$.

Consider now two reflections s_1 and s_2 with mirrors $M_1 = \{v \in \mathbb{R}^n | (v,\, a_1) = p_1\}$ and $M_2 = \{v \in \mathbb{R}^n | (v,\, a_2) = p_2\}$. What can we say about the group G generated by $s_1,\, s_2$?

Note first that G preserves all points of the orthogonal complement to $V = \mathbb{R}a_1 + \mathbb{R}a_2$. So, we can consider only this space V which has dimension 1 or 2.

Case I. M_1 is parallel to M_2, $\dim V = 1$. On the line $\mathbb{R}^1 = \mathbb{R}a_1 = \mathbb{R}a_2$ we have two mirrors $M_1,\, M_2$ which are just points. Choosing appropriate coordinate we can assume that $M_1 = 0$ and $M_2 = \frac{1}{2}$. Then $s_1 : x \mapsto -x, \quad s_2 : x \mapsto 1 - x$. The group G is the affine group $\mathrm{Aff}\,(1,\, \mathbb{Z})$ consisting of transformations $x \to ax + b$ with integer coefficients. Since the inverse transformation $x \mapsto \frac{x-b}{a}$ also must have integer coefficients, we see that $a = \pm 1$.

As an abstract group, G is a semi-direct product of the subgroup C_2 generated by s_1 and the normal subgroup $\mathbb{Z}$ generated by $s_2 s_1 : x \mapsto x + 1$.

Case II. M_1 and M_2 are not parallel, $\dim V = 2$. We can choose the origin into the intersection $M_1 \bigcap M_2$ and consider only 2-plane V spanned by a_1 and a_2. If the lines $m_i = M_i \bigcap V$ form an angle α, the transformation $s_1 s_2$ is a rotation on the angle 2α.

If α is commensurable with π, the rotation has a finite order m and the group G, as an abstract group, is isomorphic to the dihedral group D_m (the isometry group of a regular m-gon).

Otherwise, G is an infinite group isomorphic to $\mathrm{Aff}\,(1,\, \mathbb{Z})$.

Now we come to Coxeter groups. They are defined as abstract groups possessing a special set of generators and relations. Namely, let S be a finite set and assume that a function $m:\, S\times S\to\{1,\,2,\,\dots,\,\infty\}$ is given such that

$$m(s,\,s')=m(s',\,s); \qquad m(s,\,s')=1 \quad\Longleftrightarrow\quad s=s'. \tag{2}$$

A Coxeter group G corresponding to given S and m is a group with the following properties:

a) The group G is generated by the set S with the relations[1]

$$(ss')^{m(s,\,s')}=e \quad \text{for all} \quad s,\,s'\in S\times S \tag{3}$$

b) If H is any group generated by the set S with relations (3), there is a unique homomorphism $\varphi:\, G\to H$ which is identical on S.

To illustrate this definition, we prove here

Theorem. *Let $S=\{s_1,\,s_2\}$ and $m(s_1,\,s_2)=m$. Then the Coxeter group G is isomorphic to D_m if m is finite, and to Aff$(1,\,\mathbb{Z})$ if $m=\infty$.*

Proof. We consider only the case $m=\infty$. We saw already that Aff$(1,\,\mathbb{Z})$ is generated by two involutions $s_1:\, x\mapsto -x$ and $s_2:\, x\mapsto \frac{1}{2}-x$. More precisely, the element $g_{\epsilon,k}:\, x\mapsto \epsilon x+k$ can be written as:

$$(s_2s_1)^k \quad \text{if} \quad \epsilon=1, \qquad (s_2s_1)^k s_2 \quad \text{if} \quad \epsilon=-1$$

Let H be any group generated by two involutions $s_1,\,s_2$ (e.g., the group D_m). The homomorphism $\varphi:\, G\to H$ is given by

$$\varphi(g_{1,k})=(s_2s_1)^k, \qquad \varphi(g_{-1,k})=(s_2s_1)^k s_2.$$

□

[1] If $m(s,\,s')=\infty$, (3) means that the element ss' has infinite order.

Homework 1. (Due Sept. 15)

1. Give the proof of the Theorem above for a finite m.

(4 points)

2. Show that S_3 is a Coxeter group. Find S and m in this case.

(6 points)

3. Which of the following transformation groups are isomorphic as abstract groups?

a) The group of rotations of a solid cube in $\mathbb{R}^3$.

b) The group $SL(2, \mathbb{F}_3)$ of unimodular matrices over the field with 3 elements.

c) The group S_4 of permutations of four objects.

d) The group $PGL(2, \mathbb{F}_3)$ of transformations of the projective line over $\mathbb{F}_3$.

(2.5 points for each pair)

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