LECTURE 2
OPERATORS IN HILBERT SPACE

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1. Hilbert spaces

We shall consider a class of real or complex vector spaces where the notion of a self-adjoint operator makes sense. This class includes all Euclidean spaces $\mathbb{R}^n$, their complex analogues $\mathbb{C}^n$ and the classical Hilbert space H, which is infinite-dimensional complex space. All these spaces we call simply **Hilbert spaces**.

Let V be a real vector space. A map $V \times V \to \mathbb{R}$, denoted by (v_1, v_2), is called **inner product** (other terms: **scalar** or **dot**-product) if it has the following properties:

1. Positivity: $(v, v) \geq 0$ and $(v, v) = 0 \iff v = 0$.
2. Symmetry: $(v_1, v_2) = (v_2, v_1)$.
3. Linearity: $(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1(v_1, w) + \lambda_2(v_2, w)$.

For a complex vector space V the definition is almost the same with the two small corrections. First, an inner product is a map $V \times V \to \mathbb{C}$; second, the symmetry property is replaced by

2.′ Hermitian symmetry: $(v_1, v_2) = \overline{(v_2, v_1)}$, where bar means the complex conjugation.

Proposition 1. *In a finite-dimensional real (resp. complex) vector space V any inner product in an appropriate basis has the form*

$$(v, w) = \sum_k v_k w_k \qquad (\ resp. \sum_k v_k \overline{w_k}\ ). \tag{1}$$

Proof. Let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis in V. The equality (1) is equivalent to the following property of B:

$$(e_i, e_j) = \delta_{i,j} := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{2}$$

Such a basis is called **ortonormal**.

So, we have to prove that any finite-dimensional space posesses an ortonormal basis. But there is a well-known orthogonalization process which transforms any given basis into an ortonormal one.

□

Date: Sept 2007.

Using the scalar product, we can define the length of a vector $v \in V$ as

$$|v| = \sqrt{(v,\, v)}. \tag{3}$$

In the real case we define also the **angle** θ between two vectors $v,\, w$ by

$$\cos\theta \;=\; \frac{(v,\, w)}{|v|\cdot|w|}. \tag{4}$$

In the complex case the angle is not defined, but the notion of **orthogonal** or **perpendicular** vectors still makes sense. It means that their inner product vanishes.

Define the **distance** between two vectors by

$$d(v,\, w) = |v - w|. \tag{5}$$

Proposition 2. *The distance* (5) *satisfies the axioms of a metric space:*
a) Positivity: $d(v,\, w) > 0$ *and* $d(v,\, w) = 0 \iff v = w.$
b) Symmetry: $d(v,\, w) = d(w,\, v).$
c) Triangle inequality: $d(v,\, w) \;\leq\; d(v,\, u) + d(u,\, w).$

Proof. The first two properties follow immediately from the definition. The triangle inequality is equivalent to the inequality

$$|v + w| \leq |v| + |w|. \tag{6}$$

The latter follows from well-known Bunyakovski-Cauchy-Schwarz inequality:

$$|(v,\, w)| \;\leq\; |v|\cdot|w|. \tag{7}$$

□

Thus, any vector space V with an inner product is a metric space. If this metric space is complete, we call V a **Hilbert** space.

We say that a space V with an inner product is the **direct sum**, or **orthogonal sum**, of two subspaces V' and V'' and write $V = V' \oplus V''$ if

1. For any $v' \in V$ and any $v'' \in V''$ we have $(v',\, v'') = 0$. In this case we also write $V' \perp V''$.

2. Any vector $v \in V$ can be written (necessarily uniquely) in the form

$$v = v' + v'', \quad \text{where} \quad v' \in V',\; v'' \in V''.$$

Let V be a space with an inner product and $X \subset V$ be any subset. Define the **orthogonal complement** to X as

$$X^{\perp} := \{y \in H \;\big|\; (x,\, y) = 0 \quad \text{for all} \quad x \in X\}$$

It is clear that $X^{\perp}$ is a closed vector subspace in V.

Theorem 1. *Let H be a Hilbert space and $H' \subset H$ be a closed subspace. Denote by H'' the orthogonal complement ${H'}^{\perp}$. Then $H = H' \oplus H''$.*

Proof. We start with a geometric

Lemma 1. *Let H' be a closed subspace in a Hilbert space H. For any point $x \in H \backslash H'$ there is unique point $y \in H'$ which is nearest point to x. The vector $x - y$ is orthogonal to H'.*

Proof of the Lemma. Let d be the greatest lower bound for the distances $d(x, y)$ where $y \in H'$. We can find $y_n \in H'$ so that $d(x, y_n) < d + \frac{1}{n}$. Consider the parallelogram with vertices y_n, x, y_m, $y_n + y_m - x$. We have

$$2|x - y_n|^2 + 2|x - y_m|^2 = |y_n - y_m|^2 + 4\left|x - \frac{y_n + y_m}{2}\right|^2. \tag{8}$$

Since the first two lengths are $< d + \frac{1}{n}$ and the last one is $\geq d$, we obtain

$$|y_n - y_m|^2 < 4\left(d + \frac{1}{n}\right)^2 - 4d^2 = \frac{8d}{n} + \frac{4}{n^2}.$$

We see, that $d(y_n, y_m) \to 0$ when $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. But H' is closed, hence complete, and the sequence $\{y_n\}$ has a limit y. For this y we have $d(x, y) = d$.

Let now w be any vector from H'. We show that $(x - y, w) = 0$ Assume the contrary. Multiplying w by the appropriate scalar, we can assume that $(x - y, w)$ is real. Consider the function of the real variable t given by $f(t) = d(x, y + tw)^2$. By definition, this function has a minimum at $t = 0$, hence $f'(0) = 0$. On the other hand, we have $f(t) = (x - y - tw)^2 = d^2 + 2t(x - y, w) + t^2|w|^2$ and $f'(0) = (x - y, w) = 0$.

□

Come back to the theorem. Consider any $x \in H$. If $x \notin H'$, let x' be the nearest point to x in H'. If $x \in H'$, put $x' = x$. In both cases we have $x = x' + x''$ where $x' \in H'$, $x'' \in H''$.

□

Theorem 2. *Let H be a Hilbert space and $f : H \to \mathbb{C}$ be a continuous linear map. We call f a bf continuous linear functional on H. There exist a unique vector $y \in H$ such that*

$$f(x) = (x, y) \quad \textit{for all} \quad x \in H. \tag{9}$$

Proof. We can assume that $f \neq 0$ (otherwise, we could put $y = 0$). Denote by H' the kernel of the functional f, i.e. the set of vectors x such that $f(x) = 0$. It is clear that H' is a closed vector subspace in H. Let H'' be its orthogonal complement. I claim that dim $H'' = 1$. Indeed, since $f \neq 0, H' \neq H$. So, dim $H" \geq 1$. Assume that x_1 and x_2 are two vectors from H''. The vector $x := f(x_2)x_1 - f(x_1)x_2$ belongs to H'' (as a linear combination of x_1 and x_2) and to H' (because $f(x) = 0$). Hence, $x = 0$. If $f(x_1)$ is non-zero, we obtain $x_2 = -\frac{f(x_2)}{f(x_1}x_1$. If $f(x_1) = 0$, then $x_1 \in H'' \bigcap H" = \{0\}$. In both cases x_1 and x_2 are dependent. It follows that dim $H'' \leq 1$.

Thus, $\dim H'' = 1$ and $H'' = \mathbb{C} \cdot y_0$. Put $y = \frac{\overline{f(y_0)} y_0}{|y_0|^2}$. We check that $f(x) = (x,\, y)$ for all $x \in H$. For $x \in H'$ both sides are zeros. For $x \in H''$ we have $x = cy_0$. Therefore, $f(x) = cf(y_0)$ and $(x,\, y) = (cy_0,\, \frac{\overline{f(y_0)} y_0}{|y_0|^2}) = cf(y_0)$. □

Consider now an orthonormal system of vectors $\{x_\alpha\}_{\alpha \in A}$ in a Hilbert space H. We call such a system **complete** if $(\{x_\alpha\}_{\alpha \in A})^\perp = \{0\}$.

An orthonormal system $\{x_\alpha\}_{\alpha \in A}$ is called a **Hilbert basis** in H if any vector $x \in H$ can be (necessarily uniquely) written in the form

$$x = \sum_{\alpha \in A} c_\alpha x_\alpha \quad \text{where} \quad c_\alpha = (x,\, x_\alpha). \tag{10}$$

Here A can be any set of indices and we have to explain how the right hand side in (10) is defined.

Lemma 2. *For any $x \in H$ and any orthonormal system $\{x_\alpha\}_{\alpha \in A} \subset H$ only countable set of coefficiets $c_\alpha = (x,\, x_\alpha)$ can be non-zero.*

Proof. Denote by A_n the subset of those $\alpha \in A$ for which $|c_\alpha| > \frac{1}{n}$. I claim that the cardinality $|A_n|$ is finite. Indeed, for any finite subset $B \in A_n$ the vector $y = x - \sum_{\beta \in B} c_\beta x_\beta$ is orthogonal to all x_β, $\beta \in B$. Therefore $|x|^2 = |y|^2 + \sum_{\beta \in B} |c_\beta|^2$, or, $\sum_{\beta \in B} |c_\beta|^2 = |x|^2 - |y|^2 \leq |x|^2$. Since $|c_\beta| > \frac{1}{n}$ for every $\beta \in B$, we conclude that $|B| < n^2 |x|^2$. It follows that the set A_n is finite. Evidently, the union $\bigcup_{n \geq 1} A_n$ is countable and contains all indices α for which $c_\alpha \neq 0$.

□

So, we have only to define the sum of a countable family of vectors. It can be done as usual:

$$\sum_{k=1}^{\infty} c_k x_k = \lim_{n \to \infty} \sum_{k=1}^{n} c_k x_k.$$

Theorem 3. *An orthonormal system of vectors in a Hilbert space is a basis iff it is complete.*

Proof. If an orthonormal system is a Hilbet basis, then any vector, orthogonal to the system, has zero coordinates, hence is zero itself.

Let now $\{x_\alpha\}_{\alpha \in A} \subset H$ is a complete system. Show that it is a Hilbert basis. For any vector $x \in H$ consider the sum (10). According to Lemma (2), only countable set of indices c_α are non-zero. Label them by positive integers and consider the corresponding sum

$$\sum_{k=1}^{\infty} c_k x_k. \tag{11}$$

As in the proof of the lemma, we establish that $\sum_{k=1}^{\infty} |c_k|^2 \leq |x|^2$. Therefore, the remainder of the series (11) tends to zero and we denote by x' the corresponding sum. It is clear that x' has the same coordinates as x. Hence, the difference $x'-x$ is orthogonal to all x_k. It is also orthogonal to all other x_α. Since our system is complete, we get $x'=x$.

□

Now we come to examples.

The first example is the classical space l_2 of all sequences of complex numbers $\{c_k\}_{k\geq 1}$ satisfying the condition

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty \tag{12}$$

The inner product is defined by

$$(\{c_k\},\, \{b_k\}) = \sum_{k=1}^{\infty} c_k \overline{b_k}. \tag{13}$$

We leave to the reader to check the completeness of this space.

Second example is another classical space $L_2([0,1],\, dx)$ of equivalence classes of square integrable complex-valued functions on $[0,1]$. The inner product is defined by the Lebesgue integral:

$$([f],\, [g]) = \int_0^1 f(x)\overline{g(x)}dx, \qquad f \in [f],\; g \in [g]. \tag{14}$$

Actually, this space can be describe in more natural terms. Consider the space $C[0,\, 1]$ of all continuous complex-valued functions on $[0,\, 1]$ with the inner product

$$(f,\, g) = \int_0^1 f(x)\overline{g(x)}dx \tag{15}$$

given by ordinary Riemann integral. It satisfies all the axioms of Hilbert space except the completeness. It turn out that the **completion** of this space is exactly $L_2([0,1],\, dx)$. Moreover, in $C[0,\, 1]$ we can consider the subspaces $C^\infty[0,\, 1]$ of smooth functions ot $\mathrm{Pol}[0,\, 1]$ of polynomial functions or $\mathrm{Trig}[0,\, 1]$ of trigonometric polynomials. All of these subspaces are dense in $L_2([0,1],\, dx)$, hence, the latter is a completion of the former.

Theorem 4. *The system of functions*

$$e_n(x) = e^{2\pi i n x},\; n \in \mathbb{Z}, \tag{16}$$

is an orthonormal basis in the Hilbert space $H = L_2([0,1],\, dx)$.

Proof. We have to show that any element of H, which is orthogonal to all e_n is zero. We shall use the fact that this element can be arbitrary well approximated (in the Hilbert metric) by a continuous function. For the above definition of L_2 it follows from the Lebesgue theory, and from our definition by completion it is obvious. Now, we quote the Weierstrass theorem which claims that any continuous function on [0, 1] can be uniformly (hence, also in the Hilbert metric) approximated by trigonometric polynomial, i.e. by a finite linear combination of our basic functions e_n. But our element is orthogonal to this approximating functions; therefore, it is orthogonal to its limit, i.e. to itself. □

Exercise 1. *Define the functions* $B_k(x)$, $k \geq 0$, *on* $[0,1]$ *by the conditions*

$$a). B_k'(x) = kB_{k-1}(x) \quad \text{for } k \geq 1. \quad b)\, B_k(0) = B_k(1) \quad \text{for} \quad k > 1. \tag{17}$$
$$c)\, B_1(x) = x - \frac{1}{2}.$$

a) Show that B_k *is a polynomial of degree* k *with the highest term* x^k.
b) Find the coefficients of B_k *with respect to the basis* (16)
c) Express the sum $\zeta(2k) := \sum_{n\geq 1} \frac{1}{n^{2k}}$ *in terms of the constant term of* B_{2k}.

Exercise 2. *Find the angles of the triangle with vertices* $0,\ 1,\ x$ *in* $L_2([0,\ 1],\ dx)$

Exercise 3. *Let* H *be the space of holomorphic functions on* $\mathbb{C}$ *such that*

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy < \infty.$$

Show that H *is a Hilbert space with the inner product*

$$(f,\ g) = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dxdy \tag{18}$$

and a Hilbert basis

$$f_n(z) = \frac{z^n}{\sqrt{n!}},\ n \geq 0. \tag{19}$$

Exercise 4. *Find the orthonormal basis in* $L_2([-1,\ 1],\ dx)$ *by the orthogonalization of the system* $\{1,\ x,\ x^2,\ \dots\}$.

Hint Consider the Legendre polynomials $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$.

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