Course of algebra. Introduction and Problems

A.A.Kirillov *

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1 Introduction

1.1 What is Algebra?

Answer: Study of algebraic operations.

Algebraic operation: a map $M \times M \to M$.

Examples:

1. Addition $+$, subtraction $-$, multiplication $\times$, division $:$ of numbers.
2. Composition (superposition) of functions: $(f \circ g)(x) := f\big(g(x)\big)$.
3. Intersection $\cap$ and symmetric difference $\triangle$ of sets.

Unary, binary, ternary operations etc:

maps $M \to M, \quad M \times M \to M, \quad M \times M \times M \to M$...

Algebraic structure: collection of algebraic operation on a set M.

Isomorphism of algebraic structures: an algebraic structure on a set M is *isomorphic* to an algebraic structure on a set M' if there is a bijection (one-to-one correspondence) between M and N which sends any given operation on M to the corresponding operation on M'

Example:

1. The operation of addition on the set $M = \mathbb{R}$ and the operation of multiplication on the set $M' = \mathbb{R}_{>0}$.
2. The operations of additions on the set $\mathbb{Z}$ of integers and on the set $2\mathbb{Z}$ of even integers.

*IITP RAS, Bolshoy Karetny per. 19, Moscow, 127994, Russia; Department of Mathematics, The University of Pennsylvania, Philadelphia, PA 19104-6395 *E-mail address: kirillov@math.upenn.edu*

1.2 Algebra Zoo

Groups (abelian, non-abelian, subgroups, normal subgroups, quotient groups)

Rings (commutative, non-commutative, associative, non-associative, ideals, zero divisors, radical)

Modules over a ring (submodules, quotient modules, simple modules)

Fields (subfields, extensions, finite fields, simple fields, characteristic)

Vector spaces over a field (real, complex, bases, dimension, linear functionals and operators)

Algebras (real, complex, commutative, non-commutative, associative, non-associative, ideals, subalgebras and quotient algebras, simple algebras).

You will learn, during the course, the definitions and main properties of all these objects.

2 Problems

This is just a sample problems which you will learn to solve (and have to show it in homeworks and midterm and final exams).

2.1 Groups

1. Show that all non-trivial (i.e. containing more than one element) subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$ itself.

2. Show that all groups of prime order p are abelian.

3. Show that the following groups are isomorphic and establish explicit pairwise isomorphisms:

a) The group S_3 of all permutations of three objects A, B, C.

b) The group $Iso(\triangle)$ of all isometries of an equilateral triangle.

c) The group L of fractional-linear (Möbius) transformations generated by maps $x \mapsto 1 - x$ and $x \mapsto \frac{1}{x}$.

d) The group $G = GL(2, \mathbb{F}_2)$ consisting of all invertible matrices over the field $\mathbb{F}_2$.

e) The subgroup F of $PGL(2, \mathbb{Z})$, generated by matrices $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2.2 Rings

1. Let X be a set and 2^X be the collection of all subsets of X.
a) Show that 2^X is a ring with respect to the operations $\triangle$ as addition and $\cap$ as multiplication.
b) Show that 2^X is also a ring with respect to $\cap$ as addition and $\triangle$ as multiplication. Are these two ring isomorphic?
c) Are the rings above commutative? associative?

2. Establish an explicit isomorphism between the rings $\mathbb{Z}_{10}$ and $\mathbb{Z}_2 \times \mathbb{Z}_5$.
3. List all ideals in the ring $\mathbb{Z}$.

2.3 Modules

1. Let R be a commutative associative ring with a unit. Consider it as a module over itself. Describe all endomorphisms, automorphisms ant submodules of this module.

2. Show that the space $\mathbb{R}^n$ is a simple module over the matrix ring $\mathrm{Mat}_n(\mathbb{R})$.

3. Show that invertible elements of an associative ring R with a unit form a group (denoted by $R^\times$) with respect to ring multiplication.

2.4 Fields

1. Let F be a field. Find all elements $a \in F$ satisfying $a^{-1} = a$.

2. Show that there is a unique (up to an isomorphism) field $\mathbb{F}_4$ with 4 elements. This field contains, besides "mandatory" elements 0, 1, two more elements x, y.

Fill up the addition and multiplication tables below.

$$\begin{array}{c|cccc} + & 0 & 1 & x & y \\ \hline 0 & 0 & 1 & x & y \\ 1 & 1 & ? & ? & ? \\ x & x & ? & ? & ? \\ y & y & ? & ? & ? \end{array} \qquad \begin{array}{c|cccc} \times & 0 & 1 & x & y \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & x & y \\ x & 0 & x & ? & ? \\ y & 0 & y & ? & ? \end{array} \tag{1}$$

3. Does the field $\mathbb{F}_8$ contain the subfield isomorphic to $\mathbb{F}_4$?
4. Explain why there are no field of 6 elements.

2.5 Vector spaces

1. Let $v_1, v_2, \ldots, v_k$ are elements of a vector space V. Show that TFAE (the following are equivalent):
a) One of vectors $v_1, v_2, \ldots, v_k$ is a linear combination of others;
b) There exists a non-trivial linear combination of $v_1, v_2, \ldots, v_k$ equal to zero.

2. Construct bases in the following real vector spaces and determine their dimensions.
a) The space of all real rectangular matrices of size $m \times n$;
b) The space of all real symmetric square matrices of size n;
c) The space of all complex matrices of types a), b);
d) The space of all real periodic sequences $\{x_k\}_{k\in\mathbb{Z}}$ with period d (i.e. $x_{k+d} = x_k$ for all $k \in \mathbb{Z}$.)
e) The space of all real sequences $\{x_k\}_{k\in\mathbb{Z}}$ satisfying the recurrent relation

$$x_{k+n} = a_1 x_{k+n-1} + a_2 x_{k+n-2} + \cdots + a_{n-1} x_{k+1} + a_n x_k.$$

f) The space of all real-valued functions on a finite set X.

3. Let V be a 10-dimensional vector space, V' and V'' be its subspaces of dimension 7 and 8 respectively. What can be dimension of the intersection $V' \bigcap V''$?

2.6 Algebras

1. Show that the algebra $\mathrm{Mat}_n(\mathbb{R})$ has no ideals except $\{0\}$ and the algebra itself.

2. a) Show that any commutative subalgebra in $M_2(\mathbb{R})$ has dimension ≤ 2.
b*) Show that any commutative subalgebra in $M_3(\mathbb{R})$ has dimension ≤ 3.
c*) Find in $M_4(\mathbb{R})$ a commutative subalgebra of dimension 5.

3. Do the quaternions $\mathbb{H}$ form an algebra over the field $\mathbb{C}$?

4*. Classify all 2-dimensional complex associative algebras.

3 Linear algebra

A very important subject deserving more detailed discussion. (To appear)

4 Jokes, incorrect questions or expressions, problems for those who delves into every detail

1. When the system of one vector is linearly independent? What about an empty system?

2. **Warning.** The term "linear combination" is traditionally used in two different senses:

a) As an operation on vectors; as such it is determined by the collection of coefficients λ_1, λ_2, $\ldots$, λ_n.

b) As a result of this operation, i.e. the vector $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$.

Which sense is understood when we say "non-trivial linear combination" and "this combination is equal to zero"?

3. Two studends meet after classes. One said: I had an algebra class and finally understood why two lines intersect at one point. It is because the system of linear equations has a unique solution. The second replied: I had a geometry class and finally understood why the system of linear equations has a unique solution. It is because two lines intersect at one point.

The moral of the story:

First: two lines intersect at one point,

Second: the system of linear equations has a unique solution,

The last, but not least: it is the same statement.

4. **Warning.** Actually not every two lines intersect at one point and not all systems of linear equations have a unique solution. What is the correct statement?

5. Many misunderstanding is related to the expression: "reduce a matrix to such and such form". The point is that matrices have different geometric realizations, therefore the word "reduce" can imply different groups of transformations. We discuss this in details during the course.