AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Solution to Assignment No.5, December 14, 2016

1) Recall that the fair price of an European contingent claim with a $\mathcal{F}_T$-measurable payoff $f_T \geq 0$ at the time T is given by

$$V = E_{P^{\mu-r}}(e^{-rT} f_T)$$

where $E_{P^{\mu-r}}$ is the expectation with respect to the probability $P^{\mu-r}$.

(a) Let $f_T = S_T^2$ where $S_T = S_0 \exp((\mu - \frac{\sigma^2}{2})T + \sigma W_T)$ is the stock price at time T. We can write also $S_T = S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma W_T^{\mu-r})$ where $W_t^{\mu-r} = W_t + \frac{\mu-r}{\sigma} t$ is the Brownian motion with respect to the martingale measure $P^{\mu-r}$, in particular, W_T with respect to $P^{\mu-r}$ is a normal random variable with variance T and mean 0. Hence,

$$\begin{aligned} V = E_{P^{\mu-r}}(e^{-rT} S_T^2) &= S_0 e^{(r-\sigma^2)T} E_{P^{\mu-r}} \exp(2\sigma W_T^{\mu-r}) \\ &= S_0 e^{(r-\sigma^2)T} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{2\sigma x} e^{-\frac{x^2}{2T}} dx. \end{aligned}$$

Now,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{2\sigma x} e^{-\frac{x^2}{2T}} dx \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{(x-2\sigma T)^2}{2T}} e^{2\sigma^2 T} dx = e^{2\sigma^2 T}. \end{aligned}$$

Hence, $V = S_0 e^{(r+\sigma^2)T}$.

b) Now let $f_T = (S_T - K)^+$ be a call option payoff. Then

$$\begin{aligned} V = E_{P^{\mu-r}}(e^{-rT}(S_T - K)^+) &= E_{P^{\mu-r}}(\exp(-\sigma^2 T/2 + \sigma W_T^{\mu-r}) - e^{-rT} K)^+ \\ &= e^{-\sigma^2 T/2} E_{P^{\mu-r}}(\exp(\sigma W_T^{\mu-r}) - e^{(\frac{\sigma^2}{2} - r)T} K)^+ \\ &= e^{-\sigma^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (e^{\sigma x} - e^{(\frac{\sigma^2}{2} - r)T} K)^+ e^{-\frac{x^2}{2T}} dx. \end{aligned}$$

Let x_0 be such that $e^{\sigma x_0} = e^{(\frac{\sigma^2}{2} - r)T} K$ i.e. $x_0 = \sigma^{-1}((\sigma^2/2 - r)T + \ln K)$. Then

$$\begin{aligned} &\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (e^{\sigma x} - e^{(\frac{\sigma^2}{2} - r)T} K)^+ e^{-\frac{x^2}{2T}} dx \\ &= \frac{1}{\sqrt{2\pi T}} \int_{x_0}^{\infty} (e^{\sigma x} - e^{(\frac{\sigma^2}{2} - r)T} K) e^{-\frac{x^2}{2T}} dx. \end{aligned}$$

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Now,

$$\frac{1}{\sqrt{2\pi T}}\int_{x_0}^{\infty} e^{(\sigma x-\frac{x^2}{2T})}dx$$
$$=\frac{1}{\sqrt{2\pi T}}\int_{x_0}^{\infty}(e^{-\frac{(x-\sigma T)^2}{2T}}e^{\sigma^2 T/2}dx$$
$$=e^{\sigma^2 T/2}(1-\Phi(\frac{x_0-\sigma T}{\sqrt{T}})).$$

Hence,

$$V=e^{-\sigma^2 T/2}(e^{\sigma^2 T/2}(1-\Phi(\frac{x_0-\sigma T}{\sqrt{T}}))-e^{(\frac{\sigma^2}{2}-r)T}K(1-\Phi(\frac{x_0}{\sqrt{T}}))$$
$$=1-\Phi(\frac{x_0-\sigma T}{\sqrt{T}})-e^{-rT}K(1-\Phi(\frac{x_0}{\sqrt{T}}).$$

Next, for $f_T=(K-S_T)^+$ we obtain similarly

$$V=e^{-\sigma^2 T/2}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}(e^{(\frac{\sigma^2}{2}-r)T}K-e^{\sigma x})^+e^{-\frac{x^2}{2T}}dx$$
$$=e^{-\sigma^2 T/2}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{x_0}(e^{(\frac{\sigma^2}{2}-r)T}K-e^{\sigma x})^+e^{-\frac{x^2}{2T}}dx$$
$$=e^{-rT}K\Phi(\frac{x_0}{\sqrt{T}})-\Phi(\frac{x_0-\sigma T}{\sqrt{T}}).$$

2) The proof of uniqueness of the martingale measure in the Black-Scholes market was presented in the class.