AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Solution to Assignment No.4, November 30, 2016

We have a trinomial two stage market $N = 2$ with $B_n \equiv 1$, $S_n = S_0 \prod_{k=1}^{n}(1+\rho_k)$, $\rho_1, \rho_2, ...$ i.i.d., $S_0 = 1$, ρ_i equals $-\frac{1}{2}$ or $\frac{1}{2}$ or 1 each with probability $\frac{1}{3}$. We consider two payoffs $f_n = (2 - S_n)^+$-put option and $f_n = (S_n - 2)^+$-call option.

1) A probability measure $\tilde{P}$ is martingale iff $E_{\tilde{P}}\rho_i = 0$, i.e. $-\frac{1}{2}p_1 + \frac{1}{2}p_2 - p_1 - p_2 + 1 = 0$ where $p_3 = 1 - p_1 - p_2$ and $p_1 = \tilde{P}\{\rho_i = -\frac{1}{2}\}$, $p_2 = \tilde{P}\{\rho_i = \frac{1}{2}\}$ and $p_3 = \tilde{P}\{\rho_i = 1\}$. Hence, $p_2 = 2 - 3p_1$ and we have constraints $0 \leq p_1 \leq 1$, $0 \leq 2 - 3p_1 \leq 1$ and $0 \leq 1 - p_1 - p_2 \leq 1$, i.e. $0 \leq 2p_1 - 1 \leq 1$. Thus we obtain the condition $\frac{2}{3} \geq p_1 \geq \frac{1}{2}$ and p_2, p_3 are expressed by the above formulas through p_1.

2) Next, we consider the European put option. Then $E_{\tilde{P}} f_2 = E_{\tilde{P}}(2 - S_2)^+$. It is easy to see that $S_2 = \frac{1}{4}$ with probability p_1^2, $S_2 = \frac{3}{4}$ with probability $2p_1p_2$, $S_2 = \frac{9}{4}$ with probability p_2^2, $S_2 = 1$ with probability $2p_1p_3$, $S_2 = 4$ with probability p_3^2 and $S_2 = 3$ with probability $2p_2p_3$. Hence,

$$E_{\tilde{P}}(2 - S_2)^+ = \frac{7}{4}p_1^2 + \frac{1}{2}p_1p_2 + 2p_1p_3 = p_1(\frac{17}{4}p_1 - 1)$$

using the formulas above for p_2 and p_3 for any martingale measure $\tilde{P}$. Now we have to maximize this expression taking into account the constraint $\frac{1}{2} \leq p_1 \leq \frac{2}{3}$ from above. Under these constraints the above quadratic expression has the maximum at $p_1 = \frac{2}{3}$. Then $p_2 = 0$ and $p_3 = \frac{1}{3}$. Taking $\tilde{P}$ with such p_1, p_2, p_3 we obtain the superhedging price $V = E_{\tilde{P}}(2 - S_2)^+ = \frac{2}{3}(\frac{17}{6} - 1) = \frac{11}{9}$. Then we can make a computation of the self-financing hedging strategy with the initial capital V since $p_2 = 0$ and we have here, in fact, a binomial (complete) Cox-Ross-Rubinstein market where we can use the explicit martingale representation constructed in the corresponding lemma.

In the call option case:

$$E_{\tilde{P}} f_2 = E_{\tilde{P}}(S_2-2)^+ = \frac{1}{4}(2-3p_1)^2+2(2p_1-1)^2+1(2-3p_1)(2p_1-1) = \frac{17}{4}p_1^2-4p_1+1.$$

Finding the maximum of this quadratic expression in p_1 under the constraint $\frac{1}{2} \leq p_1 \leq \frac{2}{3}$ from above gives $p_1 = \frac{2}{3}$, and so $p_2 = 0$ and $p_3 = \frac{1}{3}$. Thus the superhedging price here is $V = E_{\tilde{P}}(S_2 - 2)^+ = \frac{17}{9} - \frac{8}{3} + 1 = \frac{2}{9}$. Again we arrive at a binomial (complete) Cox-Ross-Rubinstein market and we can find a corresponding self-financing hedging trading strategy with the initial capital V using the explicit martingale representation in the corresponding lemma.

3) American options case.

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For each martingale measure set

$$V_n^{\tilde{P}} = \max_{n \le \tau \le N} E_{\tilde{P}}(f_\tau | \mathcal{F}_n)$$

where the maximum is taken over finitely many stopping times (since we have here a finite probability space). Then, as we proved in the optimal stopping section,

$$V_n^{\tilde{P}} = \max(f_n,\, E_{\tilde{P}}(V_{n+1}^{\tilde{P}} | \mathcal{F}_n)),\; n = 0, 1, ..., N-1$$

with $V_N = f_N$. Since we have here $N = 2$ then $V_2^{\tilde{P}} = f_2 = (2 - S_2)^+$ in the put option case and $= (S_2 - 2)^+$ in the call option case.

Next, we deal with the put option case. Since $S_2 = (1 + \rho_2)S_1$ with ρ_2 independent of $\mathcal{F}_1$ while S_1 is measurable with respect to $\mathcal{F}_1$ then by properties of the conditional expectation and using the above formulas for p_1, p_2, p_3 corresponding to a martingale measure we obtain

$$E_{\tilde{P}}(V_2^{\tilde{P}} | \mathcal{F}_1) = (2 - \frac{1}{2}S_1)^+ p_1 + (2 - \frac{3}{2}S_1)^+(2 - 3p_1) + (2 - 2S_1)^+(2p_1 - 1).$$

Since $f_1 = (2 - S_1)^+$, $S_0 = 1$ and so $S_1 = \frac{1}{2}$ with probability p_1, $= \frac{3}{2}$ with probability $(2 - 3p_1)$ and $= 2$ with probability $(2p_1 - 1)$ we obtain the following. If $S_1 = \frac{1}{2}$ then by the above

$$E_{\tilde{P}}(V_2^{\tilde{P}} | \mathcal{F}_1) = \frac{3}{2} \text{ and } f_1 = \frac{3}{2},$$

i.e. in this case $V_1^{\tilde{P}} = \frac{3}{2}$ which happens with probability p_1. If $S_1 = \frac{3}{2}$ then

$$E_{\tilde{P}}(V_2^{\tilde{P}} | \mathcal{F}_1) = \frac{5}{4}p_1 \text{ and } f_1 = \frac{1}{2},$$

and so $V_1^{\tilde{P}} = \max(\frac{1}{2}, \frac{5}{4}p_1) = \frac{5}{4}p_1$ since $p_1 \ge \frac{1}{2}$, which happens with probability $(2 - 3p_1)$. Finally, if $S_1 = 2$ then

$$E_{\tilde{P}}(V_2^{\tilde{P}} | \mathcal{F}_1) = p_1 \text{ and } f_1 = 0,$$

and so $V_1^{\tilde{P}} = \max(0, p_1) = p_1$ which happens with probability $(2p_1 - 1)$.

Since $f_0 = 1$ and $\mathcal{F}_0$ is the trivial σ-algebra we obtain

$$V^{\tilde{P}} = V^{\tilde{P}} = \max(1, E_{\tilde{P}}(V_1^{\tilde{P}})).$$

By the above

$$E_{\tilde{P}}(V_1^{\tilde{P}}) = \frac{3}{2}p_1 + \frac{5}{4}p_1(2 - 3p_1) + p_1(2p_1 - 1) = p_1(\frac{3}{2} - \frac{7}{4}p_1).$$

Hence

$$V^{\tilde{P}} = V^{\tilde{P}} = \max(1, p_1(\frac{3}{2} - \frac{7}{4}p_1)) = 1$$

since $p_1 \le \frac{2}{3}$. Thus $V^{\tilde{P}}$ does not depend of $\tilde{P}$, and so the superhedging price V of this American put option equals 1. Then we can take $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$ and $p_3 = 0$, which yields a binomial Cox-Ross-Rubinstein market, and compute a self-financing hedging trading strategy with the initial capital equal 1 in this market.

Next, we deal with American call option. We have similarly to the above

$$E_{\tilde{P}}(V_2^{\tilde{P}} | \mathcal{F}_1) = (\frac{1}{2}S_1 - 2)^+ p_1 + (\frac{3}{2}S_1 - 2)^+(2 - 3p_1) + (2S_1 - 2)^+(2p_1 - 1) \text{ and } f_1 = (S_1 - 2)^+.$$

Now, If $S_1 = \frac{1}{2}$ then

$$E_{\tilde{P}}(V_2^{\tilde{P}} | \mathcal{F}_1) = 0 \text{ and } f_1 = 0,$$

i.e. in this case $V_1^{\tilde{P}} = 0$ which happens with probability p_1. If $S_1 = \frac{3}{2}$ then

$$E_{\tilde{P}}(V_2^{\tilde{P}}|\mathcal{F}_1) = \frac{1}{4}(2-3p_1) + (2p_1 - 1) = \frac{5}{4}p_1 - \frac{1}{2} \text{ and } f_1 = 0,$$

and so $V_1^{\tilde{P}} = \max(0, \frac{5}{4}p_1 - \frac{1}{2}) = \frac{5}{4}p_1 - \frac{1}{2}$ since $p_1 \geq \frac{1}{2}$, which happens with probability $(2-3p_1)$. Finally, if $S_1 = 2$ then

$$E_{\tilde{P}}(V_2^{\tilde{P}}|\mathcal{F}_1) = (2-3p_1) + 2(2p_1 - 1) = p_1 \text{ and } f_1 = 0,$$

and so $V_1^{\tilde{P}} = \max(0, p_1) = p_1$ which happens with probability $(2p_1 - 1)$.

Now $f_0 = 0$ and

$$E_{\tilde{P}}V_1^{\tilde{P}} = (2-3p_1)(\frac{5}{4}p_1 - \frac{1}{2}) + (2p_1 - 1)p_1 = -\frac{7}{4}p_1^2 + 3p_1 - 1.$$

Now maximizing the last expression in $p_1 \in [1/2, 2/3]$ we obtain $p_1 = \frac{2}{3}$ and the superhedging price equals $V = \frac{2}{9}$. We obtained the same result as for the European call option and this is not by chance. This is true in general since the payoff function of an American call option is a submartingale with respect to any martingale measure (check!), and so its expectation is nondecreasing function of time which means that it does not make sense to exercise such an option earlier than the expiration time (horizon) implying that both options have the same fair price (and there was no need for an additional computation for this call American option).

4) In the game option case the payoff function has the form

$$R(m,n) = (1 + (2 - S_m)^+)\mathbb{I}_{m<n} + (2 - S_n)^+\mathbb{I}_{n \leq m}$$

in the put option case and

$$R(m,n) = (1 + (S_m - 2)^+)\mathbb{I}_{m<n} + (S_n - 2)^+\mathbb{I}_{n \leq m}$$

in the call option case. Observe that there are exactly 9 stopping times between 0 and 2 in our trinomial market. For each one of them σ considered as a cancellation time by the seller we can do computations as above of the superhedging price of the American option with the payoff function $f_n = R(\sigma, n)$ and then take the minimum over these σ's.