

AN INTRODUCTION TO FINANCIAL MATHEMATICS

YURI KIFER

1. SOLUTION TO ASSIGNMENT NO.3, NOVEMBER 9, 2016

1) Since ρ_1, ρ_2, \dots are i.i.d. random variables on a probability space (Ω, \mathcal{F}, P) , the interest rate is zero $r = 0$, so that $B_n \equiv B_0$ for all n and the stock evolution is given by $S_n = S_0 \prod_{i=1}^n (1 + \rho_i)$ then P^* is a martingale measure if and only if $E_{P^*} \rho_1 = 0$. Let $A_1 = \{\rho_1 < 0\}$, $A_2 = \{0 \leq \rho_1 < a\}$ and $A_3 = \{\rho_1 \geq a\}$ which are disjoint sets (events) and we know that $P(A_1), P(A_2), P(A_3) > 0$. Define a measure P^* by

$$P^*(\Gamma) = p_1^* P\{A_1 \cap \Gamma\} + p_2^* P\{A_2 \cap \Gamma\} + p_3^* P\{A_3 \cap \Gamma\}$$

where $\infty > p_1^*, p_2^*, p_3^* > 0$ and setting $b_i = P(A_i)$, $i = 1, 2, 3$ we obtain the condition $p_1^* b_1 + p_2^* b_2 + p_3^* b_3 = 1$ which ensures that P^* is a probability measure. We obtain by the definition of P^* that for any event Γ ,

$$\max(p_1^*, p_2^*, p_3^*) P(\Gamma) \geq P^*(\Gamma) \geq \min(p_1^*, p_2^*, p_3^*) P(\Gamma),$$

and so the assumption that $\infty > p_1^*, p_2^*, p_3^* > 0$ gives that P^* is equivalent to P . In order to ensure that P^* is a martingale measure we need that

$$0 = E_{P^*} \rho_1 = p_1^* c_1 + p_2^* c_2 + p_3^* c_3$$

where $c_1 = \int_{A_1} \rho_1 dP < 0$, $c_2 = \int_{A_2} \rho_1 dP \geq 0$ and $c_3 = \int_{A_3} \rho_1 dP > 0$ since A_1, A_2, A_3 have positive probability. Thus we have two equations

$$p_1^* b_1 + p_2^* b_2 + p_3^* b_3 = 1 \quad \text{and} \quad p_1^* c_1 + p_2^* c_2 + p_3^* c_3 = 0$$

with the constraints $\infty > p_1^*, p_2^*, p_3^* > 0$. We obtain

$$p_1^* = 1 - p_2^* b_2 - p_3^* b_3 \quad \text{and so} \quad p_2^* = -\frac{c_1 + p_3^* (c_3 - c_1 b_3)}{c_2 - c_1 b_2}$$

and thus

$$p_1^* = 1 + b_2 \frac{c_1 + p_3^* (c_3 - c_1 b_3)}{c_2 - c_1 b_2} - p_3^* b_3 = \frac{c_2 + p_3^* (b_2 c_3 - b_3 c_2)}{c_2 - c_1 b_2}.$$

The denominators above are strictly positive since $c_1 < 0$ and $b_2 > 0$. If

$$0 < p_3^* < \frac{-c_1}{c_3 - c_1 b_3}$$

then $p_2^* > 0$ by the above formula. If $c_2 = 0$ then

$$p_1^* = \frac{p_3^* b_2 c_3}{-c_1 b_2} = \frac{p_3^* c_3}{-c_1} > 0$$

Date: November 14, 2016.

provided $p_3^* > 0$. If $c_2 > 0$ then $p_1^* > 0$ always if $b_2c_3 - b_3c_2 \geq 0$ while if $b_2c_3 - b_3c_2 < 0$ then $p_1^* > 0$ provided

$$p_3^* < \frac{c_2}{b_3c_2 - b_2c_3}.$$

Thus, for any positive p_3^* small enough we can find positive p_1^* and p_2^* solving two above equations, and so they admit infinitely many solutions which yield infinitely many product martingale measures $P^* = \{p_1^*, p_2^*, p_3^*\}^{\mathbb{N}}$.

2) We have here a game contingent claim in the CRR binomial market model with the horizon $N = 3$, the bond and the stock prices at time n are given by $B_n = 2^n$ and $S_n = \prod_{i=1}^n (1 + \rho_i)$, respectively, where ρ_1, ρ_2, \dots are i.i.d. random variables such that $\rho_i = 2$ with probability $1/4$ and $\rho_i = -\frac{1}{2}$ with probability $3/4$. We have here also $B_0 = S_0 = 1$ and $\rho_0 = 0$. The payoff function is given by

$$R(m, n) = Y_m \mathbb{I}_{m < n} + Z_n \mathbb{I}_{n \leq m} = (\rho_n^2 + 1) \mathbb{I}_{n \leq m} + \rho_n^2 \mathbb{I}_{m \geq n}.$$

Let P^* be the martingale measure (which is unique here) and $P^*\{\rho_1 = 2\} = p^*$. By the equation $E_{P^*}(\frac{1+\rho_1}{2}) = 1$ we have $3p^* + \frac{1}{2}(1 - p^*) = 2$, and so $p^* = \frac{3}{5}$.

By the formulas we proved the fair price of this game contingent claim is given by the formula

$$V = \min_{0 \leq \sigma \leq 3} \max_{0 \leq \tau \leq 3} E_{P^*} \left(\frac{R(\sigma, \tau)}{B_{\sigma \wedge \tau}} \right)$$

where σ, τ are stopping times and since we have here the finite probability space then we have only finitely many stopping times between 0 and 3, and so we can write min and max. On the other hand, we proved the backward induction (dynamical programming) formulas for Dynkin's game values so that if

$$V_n = \min_{n \leq \sigma \leq 3} \max_{n \leq \tau \leq 3} E_{P^*} \left(\frac{R(\sigma, \tau)}{B_{\sigma \wedge \tau}} \middle| \mathcal{F}_n \right)$$

then $V_0 = V$, $V_3 = \frac{\rho_3^2}{2^3} = \frac{\rho_3^2}{8}$ and

$$V_n = \min \left(\frac{\rho_n^2 + 1}{2^n}, \max \left(\frac{\rho_n^2}{2^n}, E_{P^*}(V_{n+1} | \mathcal{F}_n) \right) \right), \quad n = 2, 1, 0.$$

Recall that $\{\mathcal{F}_n\}$ is the filtration such that \mathcal{F}_n is generated by S_1, S_2, \dots, S_n i.e. by $\rho_1, \rho_2, \dots, \rho_n$.

Thus

$$V_2 = \min \left(\frac{\rho_2^2 + 1}{4}, \max \left(\frac{\rho_2^2}{4}, E_{P^*} \left(\frac{\rho_3^2}{8} \middle| \mathcal{F}_2 \right) \right) \right).$$

Now, by independency of ρ_3 from \mathcal{F}_2 ,

$$E_{P^*} \left(\frac{\rho_3^2}{8} \middle| \mathcal{F}_2 \right) = E_{P^*} \left(\frac{\rho_3^2}{8} \right) = \left(\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{32} \cdot \frac{2}{5} \right) = \frac{5}{16}.$$

Since $\frac{\rho_2^2 + 1}{4} \geq \frac{5}{16}$ then

$$V_2 = \max \left(\frac{\rho_2^2}{4}, \frac{5}{16} \right),$$

and so $V_2 = 1$ with P^* -probability $\frac{3}{5}$ and $V_2 = \frac{5}{16}$ with P^* -probability $\frac{2}{5}$ and V_2 is independent of \mathcal{F}_1 (since ρ_2 is independent of \mathcal{F}_1).

Now,

$$V_1 = \min \left(\frac{\rho_1^2 + 1}{2}, \max \left(\frac{\rho_1^2}{2}, E_{P^*}(V_2 | \mathcal{F}_1) \right) \right)$$

and

$$E_{P^*}(V_2|\mathcal{F}_1) = E_{P^*}(V_2) = \frac{3}{5} + \frac{1}{8} = \frac{29}{40}.$$

Next, $\max(\frac{\rho_1^2}{2}, \frac{29}{40})$ equals either 2 with P^* -probability $\frac{3}{5}$ (if $\rho_1 = 2$) or equals $\frac{29}{40}$ with P^* -probability $\frac{2}{5}$ (if $\rho_1 = -\frac{1}{2}$). Hence,

$$V_1 = \min\left(\frac{\rho_1^2 + 1}{2}, \max\left(\frac{\rho_1^2}{2}, \frac{29}{40}\right)\right)$$

and V_1 equals either 2 with P^* -probability $\frac{3}{5}$ or it equals $\frac{5}{8}$ with P^* -probability $\frac{2}{5}$.

Finally,

$$V_0 = \min(1, \max(0, E_{P^*}(V_1|\mathcal{F}_0)))$$

and $E_{P^*}(V_1|\mathcal{F}_0) = E_{P^*}(V_1) = 2 \cdot \frac{3}{5} + \frac{5}{8} \cdot \frac{2}{5} = \frac{29}{20}$. Hence, $V = V_0 = 1$. So, in fact, the seller can cancel at time 0, pay the penalty 1 which is the price of this derivative as we computed it and this is his hedging strategy with the initial capital 1.