AN INTRODUCTION TO FINANCIAL MATHEMATICS

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Abstract.

1. Lecture 9: Efficient hedging

We discussed already superhedging of derivatives in incomplete markets which is a perfect hedging since it allows the seller to build a portfolio whose value covers his/her obligation with probability one. This requirs to sell the derivative by rather high price which is often not practical. In this lecture we will discuss other possible pricing criteria which relax a perfect hedging requirement but still provide certain partial hedging which allow the seller to stay on the safe side with high high probability (quantile hedging) or by minimizing the risk that the potfolio values is not sufficient to cover the seller's obligation (shortfall risk minimization). Our discussion will follow Chapter 8 of [1].

1.1. **Quantile hedging.** Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_n\}$. Denote again by $\{X_n^\pi,\, n = 0, 1, ..., N\}$ a self-financing portfolio with a trading strategy π, let $\{\mathcal{F}_n\}$ and let f_N be $\mathcal{F}_N$-measurable adjusted payoff at time N (the horizon) for the corresponding European contingent claim (i.e. the bond price remains equal 1 all the time). Consider the following optimization problem: find a self financing trading strategy π so that

$$P\{X_N^\pi \geq f_N\} \longrightarrow \max; \text{ under the condition } X_0^\pi \leq x, \tag{1.1}$$

i.e. we want to maximize the probability that the portfolio value will be sufficient to cover the payoff at time N under the constraint that the initial capital of the portfolio does not exceed x. The corresponding problem for an American contingent claim with a payoff function $\{f_n\}$ can be written as

$$P\{X_n^\pi \geq f_n \;\; \forall n \leq N\} \longrightarrow \max; \text{ under the condition } X_0^\pi \leq x. \tag{1.2}$$

Another type of relevant optimization problems (which we will not discuss in detail) can be written as

$$x = X_0^\pi \longrightarrow \min \text{ under the condition } P\{X_N^\pi < f_N\} \leq \varepsilon, \tag{1.3}$$

i.e. we want to minimize the initial capital which ensures that probability of shortfall of the portfolio at time N does not exceed ε. The corresponding problem for American contingent claims can be written as

$$x = X_0^\pi \longrightarrow \min \text{ under the condition } P\{X_n^\pi < f_n \text{ for some } n\} < \varepsilon. \tag{1.4}$$

We consider only the European contingent claim case and consider only strategies π which do not require debt (bank loan) at the exercise time (admissible strategies),

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i.e. $X_N \geq 0$ with probability one. The event $\{X_N^\pi \geq f_N\}$ will be called "success event". We consider first the case of a complete market.

1.1. **Theorem.** *Let P^* be the unique martingale measure and assume that an event $A^* \in \mathcal{F}_N$ maximizes the probability $P(A)$ over $A \in \mathcal{F}_N$ under the condition*

$$E_{P^*}(f_N \mathbb{I}_A) \leq x. \tag{1.5}$$

Then a replicating strategy π^* for the payoff function $F^* = f_N \mathbb{I}_{A^*}$ solves the maximization problem (1.1) and the success event $\{X_N^{\pi^*} \geq f_N\}$ coincides with A^* up to an event of probability zero (i.e. the symmetric difference of these two sets has probability zero).

Proof. Let π will be any admissible self-financing trading strategy such that $X_0^\pi \leq x$. Set $A = \{X_N^\pi \geq f_N\}$. Then

$$X_N^\pi \geq f_N \mathbb{I}_A.$$

Recall, that we consider adjusted quantities, in particular, the portfolio value, and so $\{X_n^\pi\}$ is a martingale with respect to P^*. Hence,

$$E_{P^*}(f_N \mathbb{I}_A) \leq E_{P^*}(X_N^\pi) = X_0^\pi \leq x,$$

and so A satisfies (1.5). Thus

$$P(A) \leq P(A^*). \tag{1.6}$$

Now, let π^* be the replicating strategy for the payoff $f_N^* = f_N \mathbb{I}_{A^*}$. Then π^* is admissible since $X_N^{\pi^*} = f_N^* \geq 0$ and its success event satisfies

$$\{X_N^{\pi^*} \geq f_N\} = \{f_N \mathbb{I}_{A^*} \geq f_N\} \supset A^*.$$

On the other hand, by (1.6),

$$P\{X_N^{\pi^*} \geq f_N\} \leq P(A^*),$$

and so the symmetric difference of A^* and $\{X_N^{\pi^*} \geq f_N\}$ has probability zero (i.e. these events coincide up to a probability zero event). $\square$

1.2. **Definition.** Let π will be an admissible self-financing strategy. The success ratio of π is defined by

$$\psi_X = \mathbb{I}_{\{X_N^\pi \geq f_N\}} + \frac{X_N^\pi}{f_N} \mathbb{I}_{\{X_N^\pi < f_N\}}.$$

Observe that the event $\{\psi_X = 1\}$ coincides with the success event $\{X_N^\pi \geq f_N\}$.

Next, we state an optimization result in a general no arbitrage market with the set of martingale measures $\mathcal{P}(P) \neq \emptyset$. We want to find an admissible strategy π^* so that

$$E(\psi_{X^{\pi^*}}) = \sup_\pi E(\psi_{X^\pi}) \tag{1.7}$$

where the supremum is taken over all admissible strategies satisfying $X_0^\pi \leq x$ and E denotes the expectation with respect to the measure P.

1.3. **Theorem.** *There exists a measurable function $\psi^* : \Omega \to [0, 1]$ such that*

$$\sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}}(f_N \psi^*) = x < \sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}}(f_N)$$

and

$$E\psi^* = \sup_{\psi} E\psi$$

where the supremum is taken over all measurable $\psi : \Omega \to [0,1]$ *satisfying the constraint*

$$\sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}}(f_N \psi) \le x.$$

A superhedging strategy strategy π^* *for the payoff function* $f^* = f_N \psi^*$ *with the initial capital* $X_0^{\pi^*} = \sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}}(f^*)$ *solves the optimization problem (1.7).*

1.2. **Minimizing the shortfall risk.** As we saw the perfect hedging requires large initial portfolio capital

$$V = \sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}} f_N$$

where f_N is again $\mathcal{F}_N$-measurable adjusted payoff of a European contingent claim. Suppose that an investor decides to start a portfolio with a smaller sum $x \in (0, V)$ and he/she is interested to know what is the risk measured by an average portfolio shortfall defined as $E(f_N - X_N^\pi)^+$ and the problem is to choose a self-financing trading strategy π which minimizes this shortfall.

1.4. **Definition.** Let ℓ be a convex function (loss function), $\ell(x) = 0$ for $x \le 0$, ℓ is increasing on $[0, \infty)$ and $E\ell(f_N) < \infty$. The shortfall risk $r(\pi)$ of a strategy π is defined by

$$r(\pi) = E\ell(f_N - X_N^\pi) = E\ell((f_N - X_N^\pi)^+).$$

1.5. **Theorem.** *Let* ℓ *be strictly convex on* $[0, \infty)$*. Then there exists an admissible self-financing strategy* π^* *such that* $X_0^{\pi^*} \le x$ *and*

$$r(\pi^*) = \inf_{\pi} r(\pi)$$

where the infimum is taken over all admissible self-financing strategies π *satisfying* $X_0^\pi \le x$*.*

References

[1] H. Föllmer and A. Schied, *Stochastic finance*, 2nd. ed., de Gruyter, Berlin, 2004.