AN INTRODUCTION TO FINANCIAL MATHEMATICS

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Abstract.

1. Lecture 8add: Existence of a superhedging strategy for game options with the initial capital equal the superhedging price

The proof here will be a combination of the method we proved exitence equilibrium (saddle point) in Dynkins games and the technique to obtain supermartingales with respect to all martingale measures above where we replaced the supremum over all martingale measures by the supremum over martingales which are corresponding Radon-Nikodim derivatives (cf. the proof of existence of an optimal stopping time for the corresponding American contingent claim case in [1]).

We proved that the superhedging price of a game option equals

$$V=V(f,g,P)=\inf_{\sigma\in\mathcal{T}_0^N}\sup_{\tilde{P}\in\mathcal{P}(P)}\sup_{\tau\in\mathcal{T}_0^N}B_0E_{\tilde{P}}\big(\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}\big)$$

where $R(\sigma,\tau)=f_\sigma\mathbb{I}_{\sigma<\tau}+g_\tau\mathbb{I}_{\sigma\geq\tau}$. Let $\mathcal{T}_n$, $n\leq N$ be the collection of all stopping times τ satisfying $n\leq\tau\leq N$ and $\mathcal{Z}_n^N$ be the set of positive $P-martingales$ $\bar{Z}=\{\bar{Z}_k\}_{0\leq k\leq N}$ such that $\bar{Z}_0=\bar{Z}_1=\cdots=\bar{Z}_n=1$. Then

$$V=\inf_{\sigma\in\mathcal{T}_0^N}\sup_{\bar{Z}\in\mathcal{Z}_0^N}\sup_{\tau\in\mathcal{T}_0^N}B_0E\big(\bar{Z}_{\sigma\wedge\tau}\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}\big)$$

Set

$$V_n=\text{ess inf}_{\sigma\in\mathcal{T}_n^N}\text{ess sup}_{\bar{Z}\in\mathcal{Z}_n^N}\text{ess sup}_{\tau\in\mathcal{T}_n^N}B_0E\big(\bar{Z}_{\sigma\wedge\tau}\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}|\mathcal{F}_n\big).$$

Let

$$\tilde{\sigma}=\begin{cases}\sigma & \text{if}\,\sigma>n\\ N & \text{if}\,\sigma=n\end{cases}$$

which is a stopping time (check!). Then taking into account that $\bar{Z}_n=1$ when $\bar{Z}\in\mathcal{Z}_n^N$ we obtain

$$\begin{aligned}&\text{ess sup}_{\bar{Z}\in\mathcal{Z}_n^N}\text{ess sup}_{\tau\in\mathcal{T}_n^N}B_0E(\bar{Z}_{\sigma\wedge\tau}\tfrac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}|\mathcal{F}_n)\\ &=\text{ess sup}_{\bar{Z}\in\mathcal{Z}_n^N}\text{ess sup}_{\tau\in\mathcal{T}_n^N}B_0E(\bar{Z}_{\tilde{\sigma}\wedge\tau}\tfrac{R(\tilde{\sigma},\tau)}{B_{\tilde{\sigma}\wedge\tau}}|\mathcal{F}_n)\mathbb{I}_{\{\sigma>n\}}+B_0\tfrac{f_n}{B_n}\mathbb{I}_{\{\sigma=n\}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\text{ess sup}_{\bar{Z}\in\mathcal{Z}_n^N}\text{ess sup}_{\tau\in\mathcal{T}_n^N}B_0E(\bar{Z}_{\sigma\wedge\tau}\tfrac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}|\mathcal{F}_n)\\ &\geq\min(B_0\tfrac{f_n}{B_n},\text{ess sup}_{\bar{Z}\in\mathcal{Z}_n^N}\text{ess sup}_{\tau\in\mathcal{T}_n^N}B_0E(\bar{Z}_{\tilde{\sigma}\wedge\tau}\tfrac{R(\tilde{\sigma},\tau)}{B_{\tilde{\sigma}\wedge\tau}}|\mathcal{F}_n))\\ &\geq\min(B_0\tfrac{f_n}{B_n},E(V_{n+1}|\mathcal{F}_n))\end{aligned}$$

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where we used that $\mathcal{T}_{n+1}^N\subset\mathcal{T}_n^N$, $\mathcal{Z}_{n+1}^N\subset\mathcal{Z}_n^N$ and that

$$E(V_{n+1}|\mathcal{F}_n)\leq\text{ess sup}_{\bar Z\in\mathcal{Z}_{n+1}^N}\text{ess sup}_{\tau\in\mathcal{T}_{n+1}^N}B_0E(\bar Z_{\tilde\sigma\wedge\tau}\frac{R(\tilde\sigma,\tau)}{B_{\tilde\sigma\wedge\tau}}|\mathcal{F}_n).$$

The latter follows since for any $\sigma\in\mathcal{T}_{n+1}^N$,

$$E(\text{ess sup}_{\bar Z\in\mathcal{Z}_{n+1}^N}\text{ess sup}_{\tau\in\mathcal{T}_{n+1}^N}B_0E(\bar Z_{\tilde\sigma\wedge\tau}\tfrac{R(\tilde\sigma,\tau)}{B_{\tilde\sigma\wedge\tau}}|\mathcal{F}_{n+1})|\mathcal{F}_n)$$
$$\leq\text{ess sup}_{\bar Z\in\mathcal{Z}_{n+1}^N}\text{ess sup}_{\tau\in\mathcal{T}_{n+1}^N}B_0E(\bar Z_{\tilde\sigma\wedge\tau}\tfrac{R(\tilde\sigma,\tau)}{B_{\tilde\sigma\wedge\tau}}|\mathcal{F}_n).$$

This is proved similarly to to the first subsection of Lecture 8 obtaining both ess sup above as limits along nondecreasing sequences of expectations and using the monotone convergence theorem, i.e.

$$E(\lim_{k\uparrow\infty}\uparrow E(\bar Z^{(k)}_{\sigma\wedge\tau_k}\tfrac{R(\sigma,\tau_k)}{B_{\sigma\wedge\tau_k}}|\mathcal{F}_{n+1})|\mathcal{F}_n)$$
$$=\lim_{k\uparrow\infty}\uparrow E(\bar Z^{(k)}_{\sigma\wedge\tau_k}\tfrac{R(\sigma,\tau_k)}{B_{\sigma\wedge\tau_k}}|\mathcal{F}_n).$$

Thus,

$$V_n=\text{ess inf}_{\sigma\in\mathcal{T}_n^N}\text{ess sup}_{\bar Z\in\mathcal{Z}_n^N}\text{ess sup}_{\tau\in\mathcal{T}_n^N}B_0E(\bar Z_{\sigma\wedge\tau}\tfrac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}|\mathcal{F}_n)$$
$$\geq\min(B_0\tfrac{f_n}{B_n},E(V_{n+1}|\mathcal{F}_n)).$$

From the formula for V_n we have that $B_0\frac{g_n}{B_n}\leq V_n\leq B_0\frac{f_n}{B_n}$ since we always can take there $\sigma=n$ or $\tau=n$. Set

$$\sigma_0=\min\{n\leq N:\,V_n=B_0\frac{f_n}{B_n}\}.$$

On the event $\{\sigma_0>n\}$ we have that $V_n<B_0\frac{f_n}{B_n}$, and so on the event above

$$V_n\geq\min(B_0\frac{f_n}{B_n},\,E(V_{n+1}|\mathcal{F}_n))\geq E(V_{n+1}|\mathcal{F}_n).$$

It follows that

$$E(V_{\sigma_0\wedge(n+1)}|\mathcal{F}_n)=V_{\sigma_0}\mathbb{I}_{\{\sigma_0\leq n\}}+E(V_{n+1}|\mathcal{F}_n)\mathbb{I}_{\{\sigma_0>n\}}$$
$$\leq V_{\sigma_0}\mathbb{I}_{\{\sigma_0\leq n\}}+V_n\mathbb{I}_{\{\sigma_0>n\}}=V_{\sigma_0\wedge n}.$$

Hence, the sequence $V_{\sigma_0\wedge n}$ is a supermartingale, and so

$$V_0\geq EV_{\sigma_0\wedge\tau}\geq B_0E\frac{R(\sigma_0,\tau)}{B_{\sigma_0\wedge\tau}}$$

for any stopping time τ where we take into account that $\bar Z_{\sigma_0\wedge\tau}=1$ when $\bar Z\in\mathcal{Z}^N_{\sigma_0\wedge\tau}$. The latter inequality holds true since on the event $\{\sigma_0<\tau\}$ we have

$$V_{\sigma_0\wedge\tau}=V_{\sigma_0}=B_0\frac{f_{\sigma_0}}{B_{\sigma_0}}=B_0\frac{R(\sigma_0,\tau)}{B_{\sigma_0\wedge\tau}}$$

while on the event $\{\sigma_0\geq\tau\}$ we have that

$$V_{\sigma_0\wedge\tau}=V_\tau\geq B_0\frac{g_\tau}{B_\tau}=B_0\frac{R(\sigma_0,\tau)}{B_{\sigma_0\wedge\tau}}$$

since always $V_n\geq B_0\frac{g_n}{B_n}$.

Without loss of generality we can assume that P is itself a martingale measure and since no measure from $\mathcal{P}(P)$ plays a special role here we obtain that

$$V_0 \geq \sup_{\tilde{P}\in\mathcal{P}(P)} \sup_{\tau\leq N} B_0 E_{\tilde{P}}(\frac{R(\sigma_0,\tau)}{B_{\sigma_0\wedge\tau}}) = X_0^{\pi,\sigma_0}$$

where the portfolio X^{π,σ_0} was constructed before. This inequality implies that the initial capital of the hedging strategy (π, σ_0), $\pi = \pi^{\sigma_0}$ does not exceed the superhedging price V_0 and since it cannot be smaller than V_0 it equals precisely V_0. $\square$

References

[1] I. Karatzas and I.-M. Zamfirescu, *Game approach to the optimal stopping problem*, Stochastics, 401–435.