AN INTRODUCTION TO FINANCIAL MATHEMATICS

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Abstract.

1. Lecture 8: Superhedging of contingent claims in incomplete markets

1.1. **Supermartingales with respect to all martingale measures.** Let $f = (f_0, f_1, ..., f_N)$ be an adapted sequence of nonnegative random variables on a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{0\le n\le N}, P)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{P}(P)$ be a nonempty set of martingale measures. In fact, we will need for the first theorem below only the fact that all measures from $\mathcal{P}(P)$ are equivalent to P and that $\int f_k d\tilde{P} < \infty$ for each $\tilde{P} \in \mathcal{P}(P)$ and $k = 1, 2, ..., N$.

1.1. **Theorem.** *(i) Set*

$$Y_n = ess\ sup_{\tilde{P}\in\mathcal{P}(P)} E_{\tilde{P}}(f_N|\mathcal{F}_n),\ n = 0, 1, 2, ..., N,$$

where $E_{\tilde{P}}$ is the expectation with respect to $\tilde{P}$. Then $\{Y_n\}_{0\le n\le N}$ is a supermartingale with respect to each $\tilde{P} \in \mathcal{P}(P)$.

(ii) Set

$$Y_n = ess\ sup_{\tilde{P}\in\mathcal{P}(P),\, \tau\in\mathcal{T}_n^N} E_{\tilde{P}}(f_\tau|\mathcal{F}_n),\ n = 0, 1, 2, ..., N,$$

where $\mathcal{T}_n^N$ is the set of stopping times τ such that $n \le \tau \le N$. Then $\{Y_n\}_{0\le n\le N}$ is a supermartingale with respect to each $\tilde{P} \in \mathcal{P}(P)$.

Proof. We will prove only (ii). The proof of (i) is obtained in the same way as the proof of (ii) just by disregarding arguments concerning stopping times. We will use only that the measures from $\mathcal{P}(P)$ are equivalent to each other and without loss of generality we assume that $P \in \mathcal{P}(P)$. Now there is nothing special about P so we will prove without loss of generality that $Y_n, n \ge 0$ is a supermartingale with respect to P.

If $\tilde{P} \in \mathcal{P}(P)$ set

$$\tilde{Z}_N = \frac{d\tilde{P}}{dP},\ \tilde{Z}_n = \frac{d\tilde{P}_n}{dP_n},\ \tilde{Z}_0 = 1$$

where P_n and $\tilde{P}_n$ are restrictions to $\mathcal{F}_n$ of P and $\tilde{P}$, respectively. Since $\tilde{P} \sim P$ then $P\{\tilde{Z}_n > 0\} = \tilde{P}\{\tilde{Z}_n > 0\} = 1$ for all $n = 0, 1, ..., N$, and so we can define $\tilde{\rho}_n = \frac{\tilde{Z}_n}{\tilde{Z}_{n-1}}$. Then

$$\tilde{Z}_n = \prod_{k=1}^{n} \tilde{\rho}_k$$

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and, recall, that $\{\tilde Z_n\}$, $n=1,2,...$ is a martingale with respect to P. Thus, all measures $\tilde P\in\mathcal{P}(P)$ and their restrictions $\tilde P_n$ are defined by means of P and of each one of sequences $\{\tilde Z_n\}$ and $\{\tilde\rho_n\}$.

By the generalized Bayes formula for any stopping time $N\ge\tau\ge n$,

$$E_{\tilde P}(f_\tau|\mathcal{F}_n)=\frac{1}{\tilde Z_n}E_P(f_\tau\tilde Z_\tau|\mathcal{F}_n)=E_P(\tilde\rho_{n+1}\cdots\tilde\rho_\tau f_\tau|\mathcal{F}_n)$$
$$=E_P(\bar\rho_1\cdots\bar\rho_n\bar\rho_{n+1}\cdots\bar\rho_\tau f_\tau|\mathcal{F}_n)=E_P(f_\tau\bar Z_\tau|\mathcal{F}_n)$$

where $\bar\rho_1=\cdots=\bar\rho_n=1$, $\bar\rho_k=\tilde\rho_k$ for $k>n$ and $\bar Z_k=\bar\rho_1\cdots\bar\rho_k$. Thus,

$$Y_n=\text{ess sup}_{\tau\in\mathcal{T}_n^N,\,\bar Z\in\mathcal{Z}_n^N}E_P(f_\tau\bar Z_\tau|\mathcal{F}_n)$$

where $\mathcal{Z}_n^N$ is the set of positive P-martingales (martingales with respect to P) $\bar Z=\{\bar Z_k\}_{0\le k\le N}$ such that $\bar Z_0=\bar Z_1=\cdots=\bar Z_n=1$. Thus we reduced the problem from taking the supremum over a set of measures to taking the supremum over a set of martingales which is more tractable. Observe that

$$\bar{\mathcal{Z}}_n^N\subset\bar{\mathcal{Z}}_{n-1}^N\quad\text{and}\quad\mathcal{T}_n^N\subset\mathcal{T}_{n-1}^N.$$

By the basic result about esssup the above esssup can be obtained as a supremum over a sequence $\tau^{(i)}$, $\bar Z^{(i)}_{\tau^{(i)}}$ as $i\to\infty$. In fact, as it will be explained in the remark below this supremum can be obtained here along monotone non decreasing sequence, i.e.

$$\text{ess sup}_{\tau\in\mathcal{T}_n^N,\,\bar Z\in\mathcal{Z}_n^N}E_P(f_\tau\bar Z_\tau|\mathcal{F}_n)=\lim_{i\uparrow\infty}\uparrow E(f_{\tau^{(i)}}\bar Z^{(i)}_{\tau^{(i)}}|\mathcal{F}_n).$$

By the monotone convergence theorem

$$E_P(Y_n|\mathcal{F}_{n-1})=E_P\big(\text{ess sup}_{\tau\in\mathcal{T}_n^N,\,\bar Z\in\mathcal{Z}_n^N}E_P(f_\tau\bar Z_\tau|\mathcal{F}_n)|\mathcal{F}_{n-1}\big)$$
$$=E_P\big(\lim_{i\uparrow\infty}\uparrow E(f_{\tau^{(i)}}\bar Z^{(i)}_{\tau^{(i)}}|\mathcal{F}_n)|\mathcal{F}_{n-1}\big)$$
$$=\lim_{i\uparrow\infty}E_P\big(f_{\tau^{(i)}}\bar Z^{(i)}_{\tau^{(i)}}|\mathcal{F}_{n-1}\big)$$
$$\le\text{ess sup}_{\tau\in\mathcal{T}_n^N,\,\bar Z\in\mathcal{Z}_n^N}E_P\big(f_\tau\bar Z_\tau|\mathcal{F}_{n-1}\big)$$
$$\le\text{ess sup}_{\tau\in\mathcal{T}_{n-1}^N,\,\bar Z\in\mathcal{Z}_{n-1}^N}E_P\big(f_\tau\bar Z_\tau|\mathcal{F}_{n-1}\big)=Y_{n-1}$$

completing the proof that $\{Y_n\}_{0\le n\le N}$ is a supermartingale. □

1.2. **Remark.** In order to obtain a monotone limit in the above formula we used the fact that for any $\Gamma\in\mathcal{F}_n$, $\tau^{(1)},\tau^{(2)}\in\mathcal{T}_n^N$ and $\bar Z^{(1)},\bar Z^{(2)}\in\mathcal{Z}_n^N$ we can define $\tau=\tau^{(1)}\mathbb{I}_\Gamma+\tau^{(2)}\mathbb{I}_{\Omega\setminus\Gamma}\in\mathcal{T}_n^N$ and $\bar Z=\bar Z^{(1)}\mathbb{I}_\Gamma+\bar Z^{(2)}\mathbb{I}_{\Omega\setminus\Gamma}\in\mathcal{Z}_n^N$ which gives

$$E(f_\tau\bar Z_\tau|\mathcal{F}_n)=\mathbb{I}_\Gamma E(f_{\tau^{(1)}}\bar Z^{(1)}_{\tau^{(1)}}|\mathcal{F}_n)+\mathbb{I}_{\Omega\setminus\Gamma}E(f_{\tau^{(2)}}\bar Z^{(2)}_{\tau^{(2)}}|\mathcal{F}_n).$$

Choose

$$\Gamma=\{\omega:\ E(f_{\tau^{(1)}}\bar Z^{(1)}_{\tau^{(1)}}|\mathcal{F}_n)>E(f_{\tau^{(2)}}\bar Z^{(2)}_{\tau^{(2)}}|\mathcal{F}_n)\}.$$

Then

$$E(f_\tau\bar Z_\tau|\mathcal{F}_n)=\max(E(f_{\tau^{(1)}}\bar Z^{(1)}_{\tau^{(1)}}|\mathcal{F}_n),\ E(f_{\tau^{(2)}}\bar Z^{(2)}_{\tau^{(2)}}|\mathcal{F}_n)$$

which means that the family $\{E(f_\tau\bar Z_\tau|\mathcal{F}_n),\ \tau\in\mathcal{T}_n^N,\ \bar Z\in\mathcal{Z}_n^N\}$ is a lattice, and so for each sequence of random variables from this family we can construct a monotone non decreasing sequence from the same family which converges to the supremum of the former sequence.

1.2. **Superhedging of European contingent claims in incomplete markets.** We defined the fair price of a European contingent claim with a payoff f_N at time N by

$$V = V(f_N, P) = \inf\{x : \exists \pi \text{ such that } X_0^\pi = x,\, X_N^\pi \geq f_N \;\, P - \text{a.s}\}$$

where π denotes a self-financing trading strategy and X^π is the value of the corresponding portfolio.

1.3. **Theorem.** *Let a nonnegative $\mathcal{F}_N$-measurable random variable f_N satisfies*

$$\sup_{\tilde{P}\in\mathcal{P}(P)} E_{\tilde{P}} \frac{f_N}{B_N} < \infty$$

where B_N is the bond price at time N. Then

$$V = V(f_N, P) = \sup_{\tilde{P}\in\mathcal{P}(P)} B_0 E_{\tilde{P}} \frac{f_N}{B_N}$$

and there exists a hedging strategy with the initial capital V.

Proof. We already proved for the general case that for any martingal measure $\tilde{P}$ the initial capital of a self-financing hedging portfolio cannot be less than $B_0 E_{\tilde{P}} \frac{f_N}{B_N}$, and so

$$V \geq \sup_{\tilde{P}\in\mathcal{P}(P)} B_0 E_{\tilde{P}} \frac{f_N}{B_N}.$$

In order to obtain the inequality in the other direction set

$$Y_n = \text{ess sup}_{\tilde{P}\in\mathcal{P}(P)} E_{\tilde{P}}(\frac{f_N}{B_N}|\mathcal{F}_n),\; n = 0, 1, ..., N$$

which, as we proved, is a supermartingale with respect to any $\tilde{P} \in \mathcal{P}(P)$. Hence, there exists an optional decomposition

$$Y_n = Y_0 + \sum_{k=1}^{n} \gamma_k \Delta(\frac{S_k}{B_k}) - C_n,\; C_0 = 0,$$

where $\{\gamma_k\}_{1\leq k\leq N}$ is a predictable sequence and $\{C_k\}_{1\leq k\leq N}$ is an adapted non decreasing process.

Set

$$X_n^\pi = B_n(Y_0 + \sum_{k=1}^{n} \gamma_k \Delta(\frac{S_k}{B_k})),\; X_0^\pi = B_0 Y_0$$

and

$$\beta_n = \frac{1}{B_n}(X_n^\pi - \gamma_n S_n)$$

so that $X_n^\pi = \beta_n B_n + \gamma_n S_n$ and $\pi = (\beta_n, \gamma_n)_{1\leq n\leq N}$. From the definition of X_n^π above we obtain that

$$\frac{X_n^\pi}{B_n} - \frac{X_{n-1}^\pi}{B_{n-1}} = \gamma_n \Delta \frac{S_n}{B_n},$$

and so

$$\frac{X_{n-1}^\pi}{B_{n-1}} = \frac{X_n^\pi}{B_n} - \gamma_n \frac{S_n}{B_n} + \gamma_n \frac{S_{n-1}}{B_{n-1}} = \beta_n + \gamma_n \frac{S_{n-1}}{B_{n-1}}.$$

Hence, $\beta_n = \frac{X_{n-1}^\pi}{B_{n-1}} - \gamma_n \frac{S_{n-1}}{B_{n-1}}$, and so $\{\beta_n\}_{1\leq n\leq N}$ is a predictable sequence and we have also from here that

$$X_{n-1}^\pi = \beta_n B_{n-1} + \gamma_n S_{n-1}$$

imlying that π is a self-financing trading strategy. It is also hedging since

$$X_N^\pi = B_N(Y_0 + \sum_{k=1}^N \gamma_k \Delta(\frac{S_k}{B_k}))$$
$$\geq B_N(Y_0 + \sum_{k=1}^N \gamma_k \Delta(\frac{S_k}{B_k}) - C_N) = B_N Y_N = f_N$$

completing the proof. □

1.3. **Superhedging of American contingent claims.** We defined the fair price (based on hedging) of an American contingent claim with a payoff process $f = \{f_n\}_{0\leq n\leq N}$ and a horizon $N < \infty$ by

$$V = V_N(f,P) = \inf\{x : \text{there exists a self-financing strategy } \pi \text{ such that}$$
$$X_0^\pi = x,\, X_n^\pi \geq f_n\, \forall n = 1,2,...,N\, P-\text{a.s.}.$$

Let, again, $\mathcal{T}_n^N$ denotes the set of all stopping times τ with $0 \leq \tau \leq N$ and $\mathcal{P}(P)$ denotes the set of all martingale measures.

1.4. **Theorem.** *Assume that $\mathcal{P}(P) \neq \emptyset$ and let a nonnegative adapted payoff process $f = \{f_n\}_{0\leq n\leq N}$ satisfies*

$$\sup_{\tilde P\in\mathcal{P}(P)} E_{\tilde P}\frac{f_n}{B_n} < \infty \textit{ for all } n = 0,1,...,N.$$

Then

$$V = V_N(f,P) = \sup_{\tilde P\in\mathcal{P}(P),\,\tau\in\mathcal{T}_0^N} B_0 E_{\tilde P}\frac{f_\tau}{B_\tau}$$

and there exists a hedging strategy with the initial capital V.

Proof. We already proved for the general case that for any martingal measure $\tilde P$ the initial capital of a self-financing hedging portfolio cannot be less than $B_0 \sup_{\tau\in\mathcal{T}_0^N} E_{\tilde P}\frac{f_\tau}{B_\tau}$, and so

$$V \geq \sup_{\tilde P\in\mathcal{P}(P),\,\tau\in\mathcal{T}_0^N} B_0 E_{\tilde P}\frac{f_\tau}{B_\tau}.$$

In order to obtain the inequality in the other direction set

$$Y_n = \text{ess sup}_{\tilde P\in\mathcal{P}(P),\,\tau\in\mathcal{T}_n^N} E_{\tilde P}(\frac{f_\tau}{B_\tau}|\mathcal{F}_n),\ n = 0,1,...,N$$

which, as we proved, is a supermartingale with respect to any $\tilde P \in \mathcal{P}(P)$. Hence, there exists an optional decomposition

$$Y_n = Y_0 + \sum_{k=1}^n \gamma_k \Delta(\frac{S_k}{B_k}) - C_n,\ C_0 = 0,$$

where $\{\gamma_k\}_{1\leq k\leq N}$ is a predictable sequence and $\{C_k\}_{1\leq k\leq N}$ is an adapted non decreasing process.

Set, again,

$$X_n^\pi = B_n(Y_0 + \sum_{k=1}^n \gamma_k \Delta(\frac{S_k}{B_k})),\ X_0^\pi = B_0 Y_0$$

and

$$\beta_n = \frac{1}{B_n}(X_n^\pi - \gamma_n S_n)$$

so that $X_n^\pi = \beta_n B_n + \gamma_n S_n$ and $\pi = (\beta_n, \gamma_n)_{1 \leq n \leq N}$. In particular, $X_0^\pi = V_N(f, P)$. The proof that the strategy $\pi = (\beta_n, \gamma_n)_{0 \leq n \leq N}$ is self-financing, i.e. showing that $X_{n-1}^\pi = \beta_n B_{n-1} + \gamma_n S_{n-1}$, is the same as in the case of European contingent claims above since we have similar definitions here and there. The strategy π is also hedging since for all $n = 0, 1, .., N$,

$$X_n^\pi \geq B_n Y_n \geq B_n E_{\tilde{P}}(\frac{f_n}{B_n} | \mathcal{F}_n) = f_n \text{ a.s.}$$

where the first inequality follows from the definitions of Y_n, X_n^π and from the fact that $C_n \geq 0$ while the second inequality follows since we always can take the stopping time $\tau \equiv n$. The proof is complete now. □

If we can obtain a complete description of the set $\mathcal{P}(P)$ of martingale measures then in some cases the above superhedging price can be computed using the backward induction (dynamical programming) derived for optimal stopping problems.

1.5. **Corollary.** *For each martingale measure $\tilde{P}$ set $V_N^{\tilde{P}} = B_0 \frac{f_N}{B_N}$ and*

$$V_n^{\tilde{P}} = \max\big(B_0 \frac{f_0}{B_n},\, E_{\tilde{P}}(V_{n+1}^{\tilde{P}} | \mathcal{F}_n)\big)$$

for $n = N-1, N-2, ..., 1, 0$. Then the superhedging price V is given by the formula

$$V = \sup_{\tilde{P}} V_0^{\tilde{P}}.$$

1.4. **Superhedging of Israeli (game) contingent claims.** An Israeli contingent claim is determined by a payoff process $R(n,m) = f_n \mathbb{I}_{n<m} + g_m \mathbb{I}_{m \geq n}$ and a horizon $N < \infty$ where $\{f_n\}_{0 \leq n \leq N}$ and $\{g_n\}_{0 \leq n \leq N}$ are adapted sequences of random variables such that $f_n \geq g_n \geq 0$ for all $n = 0, 1, ..., n$. We defined the fair price (based on hedging) of such Israeli contingent claim by

$$V = V_N(f, g, P) = \inf\{x : \text{there exists a self-financing trading strategy } \pi \text{ and}$$
$$\text{a cancellation stopping time } \sigma \text{ such that}$$
$$X_0^\pi = x,\, X_{\sigma \wedge n}^\pi \geq R(\sigma, n)\ P-\text{a.s.} \forall n = 1, 2, ..., N.$$

Let, again, $\mathcal{T}_n^N$ denotes the set of all stopping times τ with $0 \leq \tau \leq N$ and $\mathcal{P}(P)$ denotes the set of all martingale measures.

1.6. **Theorem.** *Assume that $\mathcal{P}(P) \neq \emptyset$ and suppose that the nonnegative adapted process $f = \{f_n\}_{0 \leq n \leq N}$ satisfies*

$$\sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}} \frac{f_n}{B_n} < \infty \textit{ for all } n = 0, 1, ..., N.$$

Then

$$V = V_N(f, g, P) = \inf_{\sigma \in \mathcal{T}_0^N} \sup_{\tilde{P} \in \mathcal{P}(P),\, \tau \in \mathcal{T}_)^N} B_0 E_{\tilde{P}} \frac{R(\sigma, \tau)}{B_{\sigma \wedge \tau}}.$$

Proof. We already proved for the general case that for any martingal measure $\tilde{P}$ the initial capital of a self-financing hedging investment strategy (π, σ) cannot be less than $B_0 \sup_{\tau \in \mathcal{T}_0^N} E_{\tilde{P}} \frac{R(\sigma, \tau)}{B_{\sigma \wedge \tau}}$, and so

$$V \geq \inf_{\sigma \in \mathcal{T}_0^N} \sup_{\tilde{P} \in \mathcal{P}(P),\, \tau \in \mathcal{T}_0^N} B_0 E_{\tilde{P}} \frac{R(\sigma, \tau)}{B_{\sigma \wedge \tau}}.$$

In order to obtain the inequality in the other direction for any $\sigma \in \mathcal{T}_0^N$ set

$$Y_n^\sigma = \text{ess sup}_{\tilde{P}\in\mathcal{P}(P),\, \tau\in\mathcal{T}_n^N} E_{\tilde{P}}(\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}|\mathcal{F}_n),\ n = 0, 1, ..., N$$

which, as we proved, is a supermartingale with respect to any $\tilde{P} \in \mathcal{P}(P)$ (since $\frac{R(\sigma,n)}{B_{\sigma\wedge n}}$ is $\mathcal{F}_n$-measurable, check!). Hence, there exists an optional decomposition

$$Y_n^\sigma = Y_0^\sigma + \sum_{k=1}^n \gamma_k^\sigma \Delta(\frac{S_k}{B_k}) - C_n^\sigma,\ C_0^\sigma = 0,$$

where $\{\gamma_k^\sigma\}_{1\leq k\leq N}$ is a predictable sequence and $\{C_k^\sigma\}_{1\leq k\leq N}$ is an adapted non decreasing process.

Set

$$X_n^{\pi,\sigma} = B_n(Y_0^\sigma + \sum_{k=1}^n \gamma_k^\sigma \Delta(\frac{S_k}{B_k})),\ X_0^{\pi,\sigma} = B_0 Y_0^\sigma$$

and

$$\beta_n^\sigma = \frac{1}{B_n}(X_n^{\pi,\sigma} - \gamma_n^\sigma S_n)$$

so that $X_n^{\pi,\sigma} = \beta_n^\sigma B_n + \gamma_n^\sigma S_n$ and $\pi = \pi^\sigma = (\beta_n^\sigma, \gamma_n^\sigma)_{1\leq n\leq N}$. The proof that the strategy $\pi = (\beta_n^\sigma, \gamma_n^\sigma)_{0\leq n\leq N}$ is self-financing, i.e. showing that $X_{n-1}^{\pi,\sigma} = \beta_n^\sigma B_{n-1} + \gamma_n^\sigma S_{n-1}$, is the same as in the case of European contingent claims above since we have similar definitions here and there. The strategy (π,σ) is also hedging since for all $n = 0, 1, .., N$,

$$X_{\sigma\wedge n}^{\pi,\sigma} \geq B_{\sigma\wedge n} Y_{\sigma\wedge n}^\sigma \geq B_{\sigma\wedge n} E_{\tilde{P}}(\frac{R(\sigma,n)}{B_{\sigma\wedge n}}|\mathcal{F}_n) = R(\sigma,n) \text{ a.s.}$$

where the first inequality follows from the definitions of Y_n^σ, $X_n^{\pi,\sigma}$ and from the fact that $C_n^\sigma \geq 0$ while the second inequality follows since we always can take the stopping time $\tau \equiv n$.

Next, just by the definition of the infimum for each $\varepsilon > 0$ we can choose $\sigma_\varepsilon \in \mathcal{T}_0^N$ such that for $\pi = \pi^{\sigma_\varepsilon}$,

$$X_0^{\pi,\sigma_\varepsilon} = B_0 Y_0^{\sigma_\varepsilon} \leq V_N(f,g,P) + \varepsilon$$

which means that there exists a self-financing hedging strategy with any initial capital larger than $V_N(f,g,P)$, and so by the definition the fair price should not be bigger than $V_N(f,g,P)$ completing the proof of the theorem. □

1.7. **Remark.** We have not proved that there exists a self-financed hedging trading strategy π^* and a cancellation (stopping) time σ^* with the initial capital $X_0^{\pi^*,\sigma^*} = V_N(f,g,P)$ which does not follow directly from the theory of optimal stopping (Dynkin's) games and requires an additional argument. This can be done by combining the method we proved exitence equilibrium (saddle point) in Dynkins games and the technique to obtain supermartingales with respect to all martingale measures above where we replaced the supremum over all martingale measures by the supremum over martingales which are corresponding Radon-Nikodim derivatives (cf. the proof of existence of an optimal stopping time for the corresponding American contingent claim case in [2]).

1.5. **Pricing of contingent claims from buyer's point of view.** Define

$$W = W(f_N, P) = \sup\{x : \exists \pi \text{ such that } X_0^\pi = -x,\, X_N^\pi \geq -f_N \ P - \text{a.s}\}$$

where $f_N \geq 0$ is $\mathcal{F}_N$-measurable and $f_N > 0$ with positive probability. This quantity arises as a fair price of a European contingent claim from the following point of view. The buyer of the contract takes a bank loan x, pays for the contract and builds an investment portfolio X^π starting with the initial capital $-x$ so that at time N he/she gets the payoff f_N which covers the minus (overdraft) in his/her portfolio.

1.8. **Theorem.** *Assume that a nonnegative $\mathcal{F}_N$-measurable random variable f_N satisfies*

$$\sup_{\tilde{P} \in \mathcal{P}(P)} E_{\tilde{P}} \frac{f_N}{B_N} < \infty$$

where B_N is the bond price at time N. Then

$$W = W(f_N, P) = \inf_{\tilde{P} \in \mathcal{P}(P)} B_0 E_{\tilde{P}} \frac{f_N}{B_N}.$$

Proof. a) Assume that there exists a self-financing portfolio strategy π such that $X_0^\pi = -x$ and $X_N^\pi + f_N \geq 0$ $\tilde{P}$-a.s. Since $\{\frac{X_n}{B_n}\}_{0 \leq n \leq N}$ is a martingale with respect to $\tilde{P} \in \mathcal{P}(P)$ we obtain

$$0 \leq E_{\tilde{P}} \frac{X_N^\pi}{B_N} + E_{\tilde{P}} \frac{f_N}{B_N} = -\frac{x}{B_0} + E_{\tilde{P}} \frac{f_N}{B_N},$$

and so $x \leq B_0 E_{\tilde{P}} \frac{f_N}{B_N}$ for any $\tilde{P} \in \mathcal{P}(P)$ yilding that $x \leq W(f_N, P)$.

b) For the inequality in the other direction set

$$Y_n = \text{ess sup}_{\tilde{P} \in \mathcal{P}} E_{\tilde{P}}(-\frac{f_N}{B_N} | \mathcal{F}_n) = -\text{ess inf}_{\tilde{P} \in \mathcal{P}} E_{\tilde{P}}(\frac{f_N}{B_N} | \mathcal{F}_n).$$

Then

$$Y_0 = -\frac{W(f_N, P)}{B_0} \text{ and } Y_N = -\frac{f_N}{B_N}.$$

As we know $\{Y_n\}_{0 \leq n \leq N}$ is a supermartingale with respect to all martingale measures $\tilde{P}$, and so an optional decomposition

$$Y_n = Y_0 + \sum_{k=1}^n \gamma_k \Delta \frac{S_k}{B_k} - C_n$$

holds true.

Set

$$X_n^\pi = B_n(Y_0 + \sum_{k=1}^n \gamma_k \Delta \frac{S_k}{B_k}) \text{ and } \beta_n = \frac{X_n^\pi}{B_n} - \gamma_n \frac{S_n}{B_n} = \frac{X_{n-1}^\pi}{B_{n-1}} - \gamma_n \frac{S_{n-1}}{B_{n-1}}.$$

Then, as before, the strategy $\pi = (\beta_n, \gamma_n)_{0 \leq n \leq N}$ is self-financing, $X_0^\pi = B_0 Y_0 = -W(f_N, P)$ and

$$\frac{X_N^\pi}{B_N} \geq Y_N = -\frac{f_N}{B_N}$$

which sais that starting a portfolio with the initial capital $-W(f_N, P)$ and obtaining the payoff f_N at time N we are left with a nonnegative sum meaning that the fair price from this point of view should not be less that $W(f_N, P)$ completing the proof. □

Similar results can be obtained for American and Israeli contingent claims. The interval $(W(f_N,P),V(f_N,P))$ is called the no arbitrage interval. The reason is the following. If the contract is sold for the price $v>V(f_N,P)$ then the seller can start a hedging portfolio with the initial capital $V(f_N,P)$ and to gain the riskless profit $v-V(f_N,P)$. If the contract is sold for the price $v<W(f_N,P)$ then the buyer can take a loan $W(f_N,P)$, to have zero debt at time N and to gain the riskless profit $W(f_N,P)-v$.

On the other hand, if the selling price of the contract was $v \in (W(f_N,P),V(f_N,P))$ then there exists $\tilde P\in\mathcal{P}(P)$ such that

$$v=B_0E_{\tilde P}\frac{f_N}{B_N}>0.$$

Indeed, for any two martingale measures $\tilde P_1,\tilde P_2$ any measure $\tilde P=\alpha\tilde P_1+\beta\tilde P_2$, $\alpha,\beta\ge 0$, $\alpha+\beta=1$ is also the martingale measure. Then if X^π is a self-financing hedging investment portfolio, i.e. $X^\pi_N\ge f_N$ then

$$\frac{X^\pi_0}{B_0}=E_{\tilde P}\frac{X^\pi_N}{B_N}\ge E_{\tilde P}\frac{f_N}{B_N},$$

i.e. $X^\pi_0\ge v$. This means that in order to fulfill his/her obligation the seller has to invest into the hedging portfolio as initial capital at least the sum v obtained from the buyer and he/she gets no riskless profit. On the buyer's side, if $\frac{X^\pi_N}{B_N}+\frac{f_N}{B_N}\ge 0$ then

$$0\le E_{\tilde P}\frac{X^\pi_N}{B_N}+E_{\tilde P}\frac{f_N}{B_N}=\frac{X^\pi_0}{B_0}+E_{\tilde P}\frac{f_N}{B_N},$$

and so $X^\pi_0\ge -B_0E_{\tilde P}\frac{f_N}{B_N}=-v$. This means that if the buyer takes a loan $v'>v$ and profits from the difference $v'-v$ then any self-financing portfolio with the initial capital $-v'$ cannot cover his/her debt at time N even together with the payoff f_N, and so there is no riskless profit for the buyer, as well. Hence, v is a no arbitrage price.

References

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