AN INTRODUCTION TO FINANCIAL MATHEMATICS

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Abstract.

1. Lecture 7: Optional decomposition theorem

We start again with a probability space with a filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{0\leq n\leq N}, P)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $X = \{X_n\}_{0\leq n\leq N}$ be a (real valued) stochastic process and $S = \{S_n\}_{0\leq n\leq N}$, $S_n = (S_n^1, ..., S_n^d)$ be a $\mathbb{R}^d$-valued (d-dimensional) stochastic process, both processes adapted to the filtration $\{\mathcal{F}_n\}_{0\leq n\leq N}$. We interpret S_n as adjusted values at time n of d-stocks (so that a bond value does not appear here as it may be considered to be equal 1).

Let $\mathcal{P}(P)$ be the set of all martingale measures, i.e. the set of all probability measures equivalent to P and such that $\{S_n\}_{0\leq n\leq N}$ is a martigale with respect to each $\tilde{P} \in \mathcal{P}(P)$. Assume that $\mathcal{P}(P) \neq \emptyset$. Suppose that $X = \{X_n\}_{0\leq n\leq N}$ is a supermartingale with respect to each measure $\tilde{P} \in \mathcal{P}(P)$. If we use the Doob decomposition then we obtain that $X_n = X_0 + M_n^{(\tilde{P})} - C_n^{(\tilde{P})}$ where $M = \{M_n^{(\tilde{P})}\}_{0\leq n\leq N}$, $M_0^{(\tilde{P})} = 0$ is a martingale and $C = \{C_n^{(\tilde{P})}\}_{0\leq n\leq N,\, C_0^{(\tilde{P})}=0}$ is a predictable non decreasing process. This decomposition will depend on $\tilde{P}$ and by this reason it cannot be used for pricing of derivatives if there is more than one martingale measure. In addition, to use methods we discussed for the CRR market we would need also a martingale representation theorem which holds true only in very restricted circumstances.

1.1. **Theorem.** *(Optional decomposition theorem)(see* [2], [1]*). Suppose that $X = \{X_n\}_{0\leq n\leq N}$ is a supermartingale with respect to each measure $\tilde{P} \in \mathcal{P}(P)$. Then there exists an optional decomposition*

$$X_n = X_0 + \sum_{k=1}^{n} (\gamma_k, \Delta S_k) - C_n,\ n = 1, 2, ..., N$$

such that $\gamma = \{\gamma_k\}_{0\leq k\leq N}$ is a predictable sequence of d-dimensional random vectors and $C = \{C_k\}_{0\leq k\leq N}$ is an adapted (not predictable !) non decreasing process with $C_0 = 0$. There is no uniqueness, in general, of such a representation.

Proof. (see §2d, Chapter VI in [2]). We will prove that there exist random $\mathcal{F}_{n-1}$-measurable vectors $\gamma_n \in \mathbb{R}^d$ such that

$$\Delta X_n - (\gamma_n, \Delta S_n) \leq 0.$$

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Then we can define

$$\Delta C_n = -(\Delta X_n - (\gamma_n, \Delta S_n)) \text{ and } C_n = \sum_{k=1}^{n} \Delta C_k$$

Let P_n and $\tilde{P}_n$ be the restrictions of P and $\tilde{P}$ to the σ-algebra $\mathcal{F}_n$ and let $Z_n = \frac{d\tilde{P}_n}{dP_n}$ be the corresponding Radon-Nikodim derivatives (taking into account that these measures are equivalent). Observe that for any $\Gamma \in \mathcal{F}_{n-1}$,

$$\int_\Gamma Z_n dP = \int_\Gamma \frac{d\tilde{P}_n}{dP_n} dP_n = \tilde{P}_n(\Gamma) = \tilde{P}_{n-1}(\Gamma)$$
$$= \int_\Gamma \frac{d\tilde{P}_{n-1}}{dP_{n-1}} dP_{n-1} = \int_\Gamma \frac{d\tilde{P}_{n-1}}{dP_{n-1}} dP = \int_\Gamma Z_{n-1} dP,$$

and so $\{Z_k\}_{0\leq k\leq N}$ is a martingale. We will need a more general

1.2. **Lemma.** *(Generalized Bayes formula). Let $\mathcal{G} \subset \mathcal{H}$ be sub σ-algebras of $\mathcal{F}$ and Q, R be two equivalent probability measures. Set $Z_\mathcal{H} = \frac{dQ}{dR}|_\mathcal{H}$ and $Z_\mathcal{G} = \frac{dQ}{dR}|_\mathcal{G}$, i.e. $Z_\mathcal{H}$ and $Z_\mathcal{G}$ are $\mathcal{H}$ and $\mathcal{G}$ measurable, respectively, and $Q(A) = \int_A Z_\mathcal{H} dR$, $Q(B) = \int_B Z_\mathcal{G} dR$ for any $A \in \mathcal{H}$ and $B \in \mathcal{G}$. Then, for any $\mathcal{H}$-measurable Q-integrable random variable Y,*

$$E_Q(Y|\mathcal{G}) = \frac{1}{Z_\mathcal{G}} E_R(Y Z_\mathcal{H}|\mathcal{G}) \quad Q-(\text{or } R-) a.s.$$

where E_Q and E_R are expectations with respect to Q and R.

Proof. For any $\Gamma \in \mathcal{G}$,

$$\int_\Gamma E_Q(Y|\mathcal{G}) dQ = \int_\Gamma Y dQ = \int_\Gamma Y Z_\mathcal{H} dR = \int_\Gamma E_R(Y Z_\mathcal{H}|\mathcal{G}) dR$$
$$= \int_\Gamma \frac{1}{Z_G} E_R(Y Z_\mathcal{H}|\mathcal{G}) dQ$$

and the lemma follows. □

Using this lemma with $R = P$, $Q = \tilde{P}$, $\mathcal{G} = \mathcal{F}_m$, $\mathcal{H} = \mathcal{F}_n$, $m \leq n$ and a $\mathcal{F}_n$-measurable $\tilde{P}$-integrable random variable Y we obtain

$$\tilde{E}(Y|\mathcal{F}_m) = \frac{1}{Z_m} E(Y Z_n|\mathcal{F}_m) \;\; P-\text{a.s.}$$

where $\tilde{E}$ is the expectation with respect to $\tilde{P}$. Set $\mathcal{Z}_n = \frac{Z_n}{Z_{n-1}}$ then by the formula above

$$\tilde{E}(\Delta S_n|\mathcal{F}_{n-1}) = E(\mathcal{Z}_n \Delta S_n|\mathcal{F}_{n-1}).$$

Thus writing $\xi = \Delta X_n$ and $\eta = \Delta S_n$ we will obtain the theorem from the following general result

1.3. **Lemma.** *Let on a probability space $(\Omega, \mathcal{F}, P)$ be defined a random variable ξ and a random vector $\eta = (\eta^1, ..., \eta^d)$. Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$ and Z be a set of positive random variables $\mathcal{Z}$ such that P-a.s.,*

$$E(\mathcal{Z}|\mathcal{G}) = 1,\; E(|\xi|\mathcal{Z}|\mathcal{G}) < \infty,\; E(|\eta|\mathcal{Z}|\mathcal{G}) < \infty,\; \text{and } E(\mathcal{Z}\eta|\mathcal{G}) = 0.$$

Assume that $Z \neq \emptyset$ and that for any $\mathcal{Z} \in Z$,

$$E(\mathcal{Z}\xi|\mathcal{G}) \leq 0.$$

Then there exists a random $\mathcal{G}$-measurable vector λ^* such that

$$\xi + (\lambda^*, \eta) \leq 0 \;\; a.s.$$

Proof. We will prove the lemma for the case $\mathcal{G} = \{\Omega, \emptyset\}$ is the trivial σ-algebra. In the general case the proof is similar obtaining $\lambda^*(\omega)$ for each ω separately and then we have to use the measurable selection theorem to get a G-measurable random variable. If Ω is a finite or countable space then there is no need in the measurable selection theorem since then $\mathcal{G}$ is generated by a finite or countable partition of Ω and we can construct $\lambda^*(\omega)$ to be constant on each element of this partition in the same way as we do below in the case of the trivial $\mathcal{G}$.

Thus, we assume that $\mathcal{G} = \{\Omega, \emptyset\}$. Let $Q = Q(dx, dy)$ be the probability measure on $\mathbb{R} \times \mathbb{R}^d$ such that $Q(\Gamma) = P\{(\xi, \eta) \in \Gamma\}$ for any Borel set $\Gamma \in \mathbb{R} \times \mathbb{R}^d$. We can assume that $\eta^1, ..., \eta^d$ are linearly independent a.s., i.e. if $\sum_{i=1}^d a^i \eta^i = 0$ a.s. for some constants $a^1, ..., a^d$ then $a^1 = a^2 = ... = a^d = 0$. Indeed, for otherwise we can pass to a lower dimensional random vector η with linearly independent components. Thus, if the support of Q is in a proper subspace of $\mathbb{R}^{d+1}$ then this can only be if ξ is a linear combination of $\eta^1, ..., \eta^d$ a.s. and we obtain the result. Hence, we can we assume that the support of Q does not lie in a proper subspace of $\mathbb{R}^{d+1}$.

Let $L^0(Q)$ be the interior of the closed convex hull of the support $K(Q)$ of the measure Q. Let $x' = (x, y)$, $x \in \mathbb{R}$, $y \in \mathbb{R}^d$ and let $Z(Q)$ be the set of all Borel functions $z = z(x') > 0$ such that

$$E_Q z = 1 \quad \text{and} \quad E_Q |x'| z < \infty$$

where $E_Q v = \int v dQ$ is the expectation with respect to Q. Set

$$\Phi(Q) = \{\varphi(z) : \varphi(z) = E_Q x' z, \, z \in Z(Q)\}$$

which is the set of barycenters of probability measures Q' such that $dQ' = z dQ$. It is possible to show (see Theorem 1.48 in [1]) that $L^0(Q) = \Phi(Q)$. Observe that this is easy to understand if the samle space Ω is finite since then both ξ and η take on only finitely many values, and so Q is supported by finitely many points in $\mathbb{R}^{d+1}$. Then it is easy to see that $L^0(Q)$ and $\Phi(Q)$ are the same.

Next, we consider two cases: a) $0 \notin L^0(Q)$ and b) $0 \in L^0(Q)$. In the case a) there exists $\gamma' = (\gamma, \gamma^1, ..., \gamma^d) =\in \mathbb{R}^{d+1}$ such that

$$Q\{x' : (\gamma', x') \geq 0\} = 1 \text{ and } Q\{x' : (\gamma', x') > 0\} > 0$$

where we use that the support of Q does not belong to a proper subspace of $\mathbb{R}^{d+1}$. Hence, for these numbers $\gamma, \gamma^1, ..., \gamma^d$ a.s.,

$$\gamma\xi + (\gamma^1\eta^1 + ... + \gamma^d\eta^d) \geq 0$$

and with a positive probability

$$\gamma\xi + (\gamma^1\eta^1 + ... + \gamma^d\eta^d) > 0.$$

We claim that $\gamma \neq 0$. Indeed, if $\gamma = 0$ then $\gamma^1\eta^1 + ... + \gamma^d\eta^d \geq 0$ a.s. and by the assumption there exists $\mathcal{Z} \in Z$ such that $E\mathcal{Z}(\gamma^1\eta^1 + ... + \gamma^d\eta^d) = 0$ which implies that $\gamma^1\eta^1 + ... + \gamma^d\eta^d = 0$ a.s. contradicting the assumption that $\eta^1, ..., \eta^d$ are linearly independent unless $\gamma^1 = ... = \gamma^d = 0$ which also cannot hold true since $\gamma^1\eta^1 + ... + \gamma^d\eta^d > 0$ with positive probability.

Hence, $\gamma \neq 0$ and from the assumptions $E\mathcal{Z}\eta = 0$ and $E\mathcal{Z}\xi \leq 0$ we obtain that $\gamma < 0$. Set $\lambda^i = \frac{\gamma_i}{\gamma}$, $i = 1, ..., d$. Then $\xi + (\lambda^1\eta^1 + ... + \lambda^d\eta^d) \leq 0$, and so $\lambda^* = (\lambda^1, ..., \lambda^d)$ satisfies the conditions of this lemma completing its proof in the case a).

In the case b) we will use the Lagrange multipliers to solve a variation problem under constraints. Set $\varphi_\eta(z)=E_Qyz$ and $\varphi_\xi(z)=E_Qxz$ which are the components of the barycenter $\varphi(z)=E_Qx'z$, $x'=(x,y)$, $x\in\mathbb{R}$, $y\in\mathbb{R}^d$. Set

$$Z_0(Q)=\{z\in Z(Q):\,\varphi_\eta(z)=0\},\;\;\Phi_0(Q)=\{\varphi(z)=(\varphi_\xi(z),\varphi_\eta(z)):\,z\in Z_0(Q)\}.$$

Since $Z\neq\emptyset$ by the assumption then $Z_0(Q)\neq\emptyset$, as well, and if $z\in Z_0(Q)$ then $\varphi_\xi(z)\leq 0$. Since $0\in L^0(Q)$ then $0\in\Phi_0(Q)$, and so there exists $z_0\in Z_0(Q)$ such that $\varphi_\xi(z_0)=0$ implying that $\sup_{z\in Z_0(Q)}\varphi_\xi(z)=0$.

We proved that the supremum of $\varphi_\xi(z)$ over $z\in Z_0(Q)$ equals zero. This is equivalent to the assertion that the supremum of $\varphi_\xi(z)$ over $z\in Z(Q)$ is zero under the constraint $\varphi_\eta(z)=0$. According to the variational calculus there exists a nonzero vector λ^* such that this assertion is equivalent to the unconstraint claim that

$$\sup_{z\in Z(Q)}(\varphi_\xi(z)+(\lambda^*,\varphi_\eta(z))=0$$

(see details in §2d, Ch.4 of [2]). Hence, we obtain

$$\varphi_\xi(z)+\lambda^*\varphi_\eta(z)\leq 0\quad\forall z\in Z(Q),$$

and so

$$E_Qz(x+\lambda^*y)\leq 0\quad\forall z\in Z(Q).$$

Since the space $Z(Q)$ is large enough (it contains, in particular, all positive bounded functions z with compact support such that $E_Qz=1$) it follows from here that $x+\lambda^*y\leq 0$ Q-a.s. completing the proof. □

As explained above the theorem follows from this lemma. □

References

[1] H. Föllmer and A. Schied, *Stochastic finance*, 2nd. ed., de Gruyter, Berlin, 2004.
[2] A.N. Shiryaev, *Essentials of Stochastic Finance*, World Scientific, Singapore, 1999.