AN INTRODUCTION TO FINANCIAL MATHEMATICS

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Abstract.

1. Lecture 6: Fundamental theorems of asset pricing

1.1. **Arbitrage.** We return to the general discrete time financial market described in Lecture 4 which consists of a probability space with a filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\geq 0}, P)$ and of d-stocks $S = (S^1, ..., S^d)$, $S^i = (S^i_n)_{n\geq 0}$ and a bond $B = (B_n)_{n\geq 0}$ which will be called a general (B, S) financial market. Here

$$B_n = B_0 \prod_{1\leq k\leq n}(1 + r_k) \text{ and } S^i_n = S^i_0 \prod_{1\leq k\leq n}(1 + \rho^i_k)$$

with a predictable sequence $r_k \geq 0$, $k = 1, 2, ...$ and adapted sequences ρ^i_k, $-1 < \rho^i_k$, $k = 1, 2, ...$ We recall that a pair $\pi = (\beta, \gamma)$ of predictable sequences of random variables $\beta = \{\beta_n\}_{n\geq 0}$ and $\gamma = \{\gamma^1_n, ..., \gamma^d_n\}_{n\geq 0}$ is called a self-financing trading strategy if

$$\Delta X^\pi_n = X^\pi_n - X^\pi_{n-1} = \beta_n \Delta B_n + \sum_{i=1}^{d} \gamma^i_n \Delta S^i_n = \beta_n \Delta B_n + (\gamma_n, \Delta S_n).$$

where $X^\pi_n = \beta_n B_n + \sum_{i=1}^d \gamma^i_n S^i_n = \beta_n B_n + (\gamma_n, S_n)$ is the portfolio value at time n corresponding to the strategy π.

1.1. **Definition.** (arbitrage)

A self-financing trading strategy π provides an arbitrage opportunity at time N if $X^\pi_0 = 0$, $X^\pi_N \geq 0$ a.s. and $P\{X^\pi_N > 0\} > 0$ (equivalently, $EX^\pi_N > 0$). A financial market has no arbitrage opportunities

1) if $X^\pi_0 = 0$ and $X^\pi_N \geq 0$ implies that $X^\pi_N = 0$ a.s.;

2) (in the weak sense) if $X^\pi_0 = 0$ and $X^\pi_n \geq 0$ for all $n = 1, 2, ..., N$ implies that $X^\pi_N = 0$ a.s.;

3) (in the strong sense) if $X^\pi_0 = 0$ and $X^\pi_N \geq 0$ implies that $X^\pi_n = 0$ a.s. for all $n = 1, 2, ..., N$.

1.2. **Remark.** For studying arbitrage related issues we are talking about events of the form $\{X^\pi_n > 0\}$, $\{X^\pi_n \geq 0\}$ and $\{X^\pi_n = 0\}$, and so we can deal instead with adjusted quantities $\tilde{X}^\pi_n = \frac{X^\pi_n}{B_n}$, $\tilde{S}_n = \frac{S_n}{B_n}$ and $\tilde{B}_n = 1$ (i.e., in fact, we can assume that the bond interest rate is zero).

1.3. **Theorem.** *(First fundamental theorem of asset pricing) The general (B, S) financial market defined above has no arbitrage opportunities (in the sense of 1))*

Date: October 19, 2016.

if and only if there exists a martingale measure (i.e. a measure $\tilde{P}$ *equivalent to* P *such that* $\tilde{S}_n = \frac{S_n}{B_n}$, $n \geq 0$ *is a martingal with respect to it).*

Proof. a) Suppose that a martingale measure $\tilde{P}$ exists, π is a self-financing strategy and X^π_n is the value of the corresponding portfolio. Then $\tilde{X}^\pi_n = \frac{X^\pi_n}{B_n}$, $n \geq 0$ is a martingale with respect to $\tilde{P}$. Hence, if $X^\pi_0 = 0$ a.s. then $0 = E\tilde{X}^\pi_0 = E\tilde{X}^\pi_N$, and so if $\tilde{X}^\pi_N \geq 0$ a.s. then $\tilde{X}^\pi_N = 0$, i.e. there is no arbitrage opportunities.

b) The other direction (no arbitrage opportunities implies existence of a martingale measure) is more difficult to prove. For a proof in full generality see [1] and most of the details can also be found in [3] (see also [2]). In class we will discuss a simpler proof in a particular case of a finite probability space which can be found in Chapter 3 of [4]. □

1.4. **Remark.** (i) In the CRR market we saw that a martingale measure exists and it is unique;

(ii) If $d = \infty$ (infinitely many stocks) then the theorem is not valid since there exists a market without arbitrage opportunities and without martingale measures (see [3], p.415).

(iii) If $N = \infty$ (infinite horizon, perpetual market securities) then the theorem is not valid again since in this case there exists a market with arbitrage opportunities and with a martingale measure.

1.2. **Complete and incomplete markets.**

1.5. **Definition.** A financial market defined above on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ and a horizon N is called complete (or N-complete) if for any $\mathcal{F}_N$-measurable payoff function $f_N = f_N(\omega)$ such that $E|\frac{f_N}{B_N}| < \infty$ there exists a self-financing trading strategy π with an initial capital x so that the corresponding portfolio value satisfies $X^\pi_0 = x$ and $X^\pi_N = f_N$ i.e. any (European) contingent claim is replicable (or attainable). Otherwise the market is called incomplete.

Denote by $\mathcal{P}_N(P)$ the set of all martingale measures for the above maket with a horizon N..

1.6. **Theorem.** *(Second fundamental theorem of asset pricing). A financial market defined above without opportunity of arbitrage and with* $d, N < \infty$ *is complete if and only if* $\mathcal{P}_N(P)$ *consists of one measure only.*

Proof. a) Assume that the market is complete. Let $\Gamma \in \mathcal{F}_N$ and define $f_N(\omega) = \mathbb{I}_\Gamma(\omega)$. By completness of the market there exists a self-financing trading strategy π and an initial capital x such that $X^\pi_0 = x$ and $X^\pi_N = f_N$. If there exist two martingale measures P_1 and P_2 then $\left.\frac{X^\pi_n}{B_n}\right._{n\geq 0}$ is a martingale with respect to both P_1 and P_2. Then for $i = 1, 2$,

$$\frac{x}{B_0} = \frac{X^\pi_0}{B_0} = E_{P_i}\frac{X^\pi_N}{B_N} = E_{P_i}\frac{\mathbb{I}_\Gamma}{B_N} = \int_\Gamma B_N^{-1} dP_i.$$

Hence, $\int_\Gamma B_N^{-1} dP_1 = \int_\Gamma B_N^{-1} dP_2$ for any $\Gamma \in \mathcal{F}_N$ and since $B_N > 0$ we obtain that $P_1 = P_2$ (check!)

b) It is more difficult to prove the other direction: if there exists only one martingale measure then any payoff function with the above integrability condition is attainable. For the full proof we refer to [1] and most of the details can be found in [3] (see also [2]). In class we will discuss a simpler proof for the particular case of a finite probability space which can be found in Chapter 3 of [4].

□

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