AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Lecture 5: Derivatives in Cox-Ross-Rubinstein (CRR) market model

1.1. **Cox-Ross-Rubinstein (CRR) (binomial) market model.** In this model the market acts on a probability space $(\Omega, \mathcal{F}, P)$ and it consists of two securities:

a) a bond with the price evolution $B_n = B_0(1+r)^n$ where $r \geq 0$ is a constant (interest rate) and

b) a stock with the price evolution $S_n = S_0 \prod_{1 \leq k \leq n}(1+\rho_k)$ where $\rho_1, \rho_2, ...$ are i.i.d. random variables taking on only values so that

$$\rho_k = \begin{cases} b & \text{with probability } p \\ a & \text{with probability } 1-p \end{cases}$$

where $0 < p < 1$. In addition, we assume that $b > r > a > -1$ since if this does not hold then the model becomes trivial and not interesting. Indeed, if $r \geq b$ then it does not make sense to buy a stock since investing all money in a bond yields the riskless maximal profit. If $a \geq r$ then it is best to invest all money in the stock and the bond becomes useless. We usually assume also that $a < 0$ so that investing in the stock may yield both profit and loss. The assumption $a > -1$ means that the stock price remains positive though it may become arbitrarily small.

It is assumed also that there is a horizon N so that the market is active at times $n = 0, 1, ..., N$. We consider the filtration $\{\mathcal{F}_n\}_{0 \leq n \leq N}$ generated by the stock prices process $\mathcal{F}_n = \sigma\{S_0, S_1, ..., S_n\}$ where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ is the trivial σ-algebra, and so S_0 is a constant, while we assume that $\mathcal{F}_N = \mathcal{F}$, i.e. all our information comes from the evolution of stock prices and $\mathcal{F}_n$ is interpreted as information market participants have up to time n (inclusive). Set $\varepsilon_n = (2\rho_n - a - b)(b-a)^{-1}$. Then $\varepsilon_1, \varepsilon_2, ..., \varepsilon_N$ are i.i.d. random variables and

$$\varepsilon_k = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p. \end{cases}$$

Then $\rho_n = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\varepsilon_n$, and so each sequence $\varepsilon_1, ..., \varepsilon_n$, $\rho_1, ..., \rho_n$ and $S_1, ..., S_n$ determine uniquely each other. It follows, that in order to describe randomness generated by the stock evolution it suffices to consider the product space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{-1, 1\}^N = \{\omega = (\omega_1, \omega_2, ..., \omega_N),\ \omega_i = 1 \text{ or } = -1\}$ and $P = \{p, 1-p\}^N$, so that for $\omega = (\omega_1, ..., \omega_N)$,

$$P(\omega) = p^{\frac{1}{2}\sum_{i=1}^N(\omega_i+1)}(1-p)^{N - \frac{1}{2}\sum_{i=1}^N(\omega_i+1)}.$$

The σ-algebra $\mathcal{F}$ here is just the (finite) collection of all subsets of the space (of sequences) Ω.

Date: November 15, 2016.

1.1. **Theorem.** *In the market model above there exists a unique martingale measure* $P^* = \{p^*, 1 - p^*\}^N$ *where* $p^* = \frac{r-a}{b-a}$.

Proof. Denote by E^* expectation with respect to the probability P^* then

$$E^*(\tfrac{S_n}{B_n}|\mathcal{F}_{n-1}) = \tfrac{S_{n-1}}{B_{n-1}} E^* \tfrac{1+\rho_n}{1+r}$$
$$= \tfrac{S_{n-1}}{B_{n-1}}(1+r)^{-1}(p*(1+b)+(1-p^*)(1+a)) = \tfrac{S_{n-1}}{B_{n-1}}$$

where we use that S_{n-1} is $\mathcal{F}_{n-1}$-measurable while ρ_n is independent of the σ-algebra $\mathcal{F}_{n-1}$. Hence, the sequence $\{\frac{S_n}{B_n}, n = 0, 1, ..., N\}$ is a martingale with respect to the probability measure P^*, and so the latter is a martingale measure. In fact, this is true also when $N = \infty$ taking P^* to be the product measure on the space of infinite sequences defined on cylinder sets determined by finite sequences by the above formula. Nevertheless, we will consider usually $N < \infty$.

Next, we will prove the uniqueness. Let Q be another martingale probability measure on $(\Omega, \mathcal{F})$ with $\mathcal{F} = \mathcal{F}_N$ and let E_Q be the expectation with respect to Q. If

$$E_Q(\frac{S_n}{B_n}|\mathcal{F}_{n-1}) = \frac{S_{n-1}}{B_{n-1}}$$

then

$$E_Q(\frac{1+\rho_n}{1+r}|\mathcal{F}_{n-1}) = 1 \text{ and so } E_Q(\rho_n|\mathcal{F}_{n-1}) = r.$$

It follows that

$$Q\{\rho_n = a|\mathcal{F}_{n-1}\}a + Q\{\rho_n = b|\mathcal{F}_{n-1}\}b = r.$$

Since, $Q\{\rho_n = a|\mathcal{F}_{n-1}\} + Q\{\rho_n = b|\mathcal{F}_{n-1}\} = 1$ we obtain that

$$Q\{\rho_n = b|\mathcal{F}_{n-1}\} = \frac{r-a}{b-a} \text{ and } Q\{\rho_n = a|\mathcal{F}_{n-1}\} = \frac{b-r}{b-a}.$$

Hence, we have a random variable $X = \rho_n$ such that for any Borel set Γ the conditional probability

$$Q\{X \in \Gamma|\mathcal{G}\} = E_Q(\mathbb{I}_{\{X\in\Gamma\}}|\mathcal{G}),\ \mathcal{G} = \mathcal{F}_{n-1}$$

is constant Q-almost surely. Hence,

$$E_Q(\mathbb{I}_{\{X\in\Gamma\}}|\mathcal{G}) = Q\{X \in \Gamma|\mathcal{G}\} = Q\{X \in \Gamma\} = E_Q(\mathbb{I}_{\{X\in\Gamma\}})\ Q-\text{a.s.},$$

and so for any $A \in \mathcal{G}$,

$$Q(A \cap \{X \in \Gamma\}) = \textstyle\int_A \mathbb{I}_\Gamma(X)dQ = \int_A E_Q(\mathbb{I}_{\{X\in\Gamma\}}|\mathcal{G})dQ$$
$$= \textstyle\int_A Q\{X \in \Gamma\}dQ = Q(A)Q\{X \in \Gamma\}.$$

Hence, A and $\{X \in \Gamma\}$ are independent and this being true for all $A \in \mathcal{G}$ and any Borel Γ means that X is independent of the σ-algebra $\mathcal{G}$.

Applying this to our situation we conclude that $\rho_1, \rho_2, ..., \rho_N$ are independent with respect to Q (since each $\mathcal{F}_k$ is generated by $\rho_1, ..., \rho_k$ by the definition) and $Q\{\rho_k = b\} = \frac{b-r}{b-a} = 1 - Q\{\rho_k = a\}$ for all $k = 1, ..., N$. It follows that on the sequence space Ω described above Q coincides with P^*. $\square$

It will be important for what follows to understand that if we consider a multinomial instead of binomial model then there are already infinitely many martingale measures. Namely, let $\rho_1, \rho_2, ..., \rho_N$ be i.i.d. random variables such that $\rho_1 = a_j$ with probability p_j, $j = 1, 2, ..., m$ with $m \geq 3$, $a_1 < a_2 < ... < a_N$, $p_j \geq 0$ and $p_1 + p_2 + \cdots + p_m = 1$. Now, we consider the product (sequence) space

$(\Omega, \mathcal{F}, P)$ where $\Omega = \{1, 2, ..., m\}^N = \{\omega = (\omega_1, \omega_2, ..., \omega_N)$ where each ω_j takes values $1, 2, ..., m$ and $P = \{p_1, p_2, ..., p_m\}^N$. Then P will be a martingale measure if and only if

$$E_P \frac{1+\rho_1}{1+r} = 1, \text{ i.e. } E_P \rho_1 = r$$

which means that

$$\sum_{k=1}^{m} p_k a_k = r.$$

Here we have one equation for m variables $p_1, ..., p_m$ to which we have to add another equation $p_1 + \cdots + p_m = 1$ and the condition $p_j \geq 0$ for all $j = 1, ..., m$. It is easy to see that if $\min_j a_j < r < \max_j a_j$ these equations have infinitely many nonnegative solutions in $p_1, ..., p_m$ (check!).

1.2. **A martingale representation lemma.** Consider again the product probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{-1, 1\}^N = \{\omega = (\omega_1, \omega_2, ..., \omega_N),\ \omega_i = 1 \text{ or } = -1\}$ and $P = \{p, 1-p\}^N$ together with the i.i.d. random variables $\varepsilon_1, \varepsilon_2, ..., \varepsilon_N$ given by $\varepsilon_k(\omega) = \omega_k$ for each $\omega = (\omega_1, \omega_2, ..., \omega_N)$ so that $\varepsilon_k = 1$ with probability p and $\varepsilon_k = -1$ with probability $1-p$. As before we consider also the filtration $\{\mathcal{F}_k\}_{0 \leq k \leq N}$ where $\mathcal{F}_k = \sigma\{\varepsilon_1, ..., \varepsilon_k\}$ for $k \geq 1$, $\mathcal{F}_N = \mathcal{F}$ and $\mathcal{F}_0$ is a trivial σ-algebra.

1.2. **Lemma.** *Let* $M = \{M_n\}_{0 \leq n \leq N}$ *be a martingale on the probability space* $(\Omega, \mathcal{F}, P)$ *with respect to the filtration* $\{\mathcal{F}_n\}_{0 \leq n \leq N}$. *Then there exists a unique predictable sequence* $\{H_n\}_{1 \leq n \leq N}$ *such that*

$$M_n = M_0 + \sum_{k=1}^{n} H_k(\varepsilon_k - 2p + 1) \quad \textit{for } n = 1, 2, ..., N.$$

Proof. Since M_n is $\mathcal{F}_n$-measurable and the latter is generated by $\varepsilon_1, ..., \varepsilon_n$ then $M_n(\omega) = f_n(\varepsilon_1(\omega), ..., \varepsilon_n(\omega)) = f_n(\omega_1, ..., \omega_m)$ for some function $f_n : \{-1, 1\}^n \to \mathbb{R}$. (Here it is trivial but prove that in general: if X is a random vector and Y is a random variable measurable with respect to $\sigma(X)$ then $Y = f(X)$ for some Borel function f).

Now, $M_k, k \geq 0$ is a martingale, and so

$$\begin{aligned} 0 &= E(M_n - M_{n-1} | \mathcal{F}_{n-1})(\omega) \\ &= p f_n(\omega_1, ..., \omega_{n-1}, 1) + (1-p) f_n(\omega_1, ..., \omega_{n-1}, -1) - f_{n-1}(\omega_1, ..., \omega_{n-1}) \end{aligned}$$

where we used that ε_n is independent of $\mathcal{F}_{n-1}$ while $\varepsilon_1, ..., \varepsilon_{n-1}$ are measurable with respect to it. In view of this equality we can define

$$\begin{aligned} H_n(\omega) &= \tfrac{f_n(\omega_1, ..., \omega_{n-1}, 1) - f_{n-1}(\omega_1, ..., \omega_{n-1})}{2(1-p)} \\ &= \tfrac{f_{n-1}(\omega_1, ..., \omega_{n-1}) - f_n(\omega_1, ..., \omega_{n-1}, -1)}{2p}. \end{aligned}$$

Then H_n is $\mathcal{F}_{n-1}$-measurable and we have to show that

$$M_n = M_0 + \sum_{k=1}^{n} H_k(\varepsilon_k - 2p + 1).$$

This equality holds true trivially for $n = 0$. Suppose that it holds true for all n up to $m - 1$. Then

$$M_m(\omega) = f_m(\omega_1, ..., \omega_m) =$$

$$= \begin{cases} \dfrac{f_m(\omega_1,...,\omega_{m-1},1)-f_{m-1}(\omega_1,...,\omega_{m-1})}{2(1-p)}(\omega_m-2p+1)+f_{m-1}(\omega_1,...,\omega_m) \\ \text{if } \omega_m=1 \\ -\dfrac{f_m(\omega_1,...,\omega_{m-1},-1)-f_{m-1}(\omega_1,...,\omega_{m-1})}{2p}(\omega_m-2p+1)+f_{m-1}(\omega_1,...,\omega_m) \\ \text{if } \omega_m=-1. \end{cases}$$

$$= H_m(\omega)(\varepsilon_m(\omega)-2p+1)+M_{m-1}(\omega)$$

completing the proof by induction.

It remains only to establish the uniqueness. Suppose that the representation holds true also for $H'_k, k=1,...,N$. then

$$\sum_{k=1}^n (H_k-H'_k)(\varepsilon_k-2p+1)=0.$$

Then

$$0=E\big(\sum_{k=1}^n (H_k-H'_k)(\varepsilon_k-2p+1)|\mathcal{F}_1\big)=(H_1-H'_1)(\varepsilon_1-2p+1)$$

implying that $H_1=H'_1$ since $\varepsilon_1-2p+1\neq 0$ and we used that $E\big((H_k-H'_k)(\varepsilon_k-2p+1)|\mathcal{F}_{k-1}\big)=0$ for each k. Now assume that $H_k=H'_k$ for all $k\le m-1$. Then

$$0=E\big(\sum_{k=m}^n (H_k-H'_k)(\varepsilon_k-2p+1)|\mathcal{F}_m\big)=(H_m-H'_m)(\varepsilon_m-2p+1)$$

implying that $H_m=H'_m$ completing the proof by induction. □

1.3. Fair price of options in CRR market.

1.3. **Theorem.** *Let $P^*=\{p^*,1-p*\}^N$, $p^*=\frac{r-a}{b-a}$ be the martingale measure in the CRR market and denote by E^* the expectation with respect to P^*. Then the fair price V of a contingent claim with a payoff process R is given by*

(i) in the European case:

$$V=B_0E^*(\frac{R_N}{B_N});$$

(ii) in the American case:

$$V=\sup_{0\le\tau\le N} B_0E^*(\frac{R_\tau}{B_\tau}),$$

where the supremum is taken over the stopping times;

(iii) in the Israeli (game) case

$$V=\inf_{0\le\sigma\le N}\sup_{0\le\tau\le N} B_0E^*(\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}),$$

where both the infimum and the supremum are taken over the stopping times.

Proof. (i) We already obtained the appropriate lower bound even in the case of a more general discrete time market, and so it remains only to show that there exists a hedging self-financing portfolio strategy with the initial capital

$$x^*=B_0E^*(\frac{R_N}{B_N}).$$

Introduce the martingale

$$M_n = E^*(B_0 \frac{R_N}{B_N} | \mathcal{F}_n),\ n = 0, 1, 2, ...$$

Since $\varepsilon_k - 2p^* + 1 = 2(\rho_k - r)(b-a)^{-1}$ we obtain by the martingale respresentation lemma for the CRR market that

$$\begin{aligned} M_n &= M_0 + \textstyle\sum_{k=1}^n (1+r)^{-k} \gamma_k S_{k-1} \frac{1}{2}(b-a)(\varepsilon_k - 2p^* + 1) \\ &= M_0 + \textstyle\sum_{k=1}^n (1+r)^{-k} \gamma_k S_{k-1}(\rho_k - r) \end{aligned}$$

where $(1+r)^{-k} = B_0 B_k^{-1}$ and γ_k, $k \geq 1$ is a predictable sequence obtained from the martingale representation lemma so that $\gamma_k = 2H_k(1+r)^k((b-a)S_{k-1})^{-1}$. Set $X_n = (1+r)^n M_n$ and $\beta_n = (X_{n-1} - \gamma_n S_{n-1})B_{n-1}^{-1}$ so that

$$X_{n-1} = \beta_n B_{n-1} + \gamma_n S_{n-1}.$$

Observe that (β_n, γ_n), $n \geq 1$ is a predictable sequence and in order to prove that it is a self-financing portfolio strategy we have to show that

$$X_n = \beta_n B_n + \gamma_n S_n,\ n = 1, 2, ...$$

By the martingale representation and the formula for X_{n-1} we obtain

$$\begin{gathered} X_n = (1+r)^n M_n = (1+r)^n M_{n-1} + \gamma_n S_{n-1}(\rho_n - r) \\ = (1+r)X_{n-1} + \gamma_n S_{n-1}(\rho_n - r) = (1+r)(\beta_n B_{n-1} + \gamma_n S_{n-1}) \\ + \gamma_n S_{n-1}(\rho_n - r) = \beta_n B_n + \gamma_n(1+\rho_n)S_{n-1} = \beta_n B_n + \gamma_n S_n, \end{gathered}$$

and so $\pi = (\beta_n, \gamma_n)$, $n \geq 1$ is a self-financing portfolio strategy. Finally, $X_0 = M_0 = x^*$ and $X_N = (1+r)^N = R_N$, and so the strategy π with the initial capital x^* is hedging. It follows that the fair price V should not be bigger than x^* which together with the estimate in the other direction obtained earlier yields that $V = x^*$.

(ii) In the American contingent claim case we define

$$Y_n = \max_{n \leq \tau \leq N} E^*(\frac{B_0}{B_\tau} R_\tau | \mathcal{F}_n)$$

(we can take max here since there are only finitely many stopping times between 0 and N on a finite probability space Ω). As we proved it in the optimal stopping section the sequence $\{Y_n,\ n = 0, 1, ..., N\}$ is a supermartingale and by the Doob supermartingale decomposition theorem

$$Y_n = M_n - A_n,\ \ n = 0, 1, ..., N$$

where M_n, $n \geq 0$ is a martingale, $M_0 = Y_0$ and A_n, $n \geq 0$, $A_0 = 0$ is a non decreasing predictable process.

Again, we use the martingale respresentation lemma for the CRR market to obtain

$$\begin{aligned} M_n &= M_0 + \textstyle\sum_{k=1}^n (1+r)^{-k} \gamma_k S_{k-1} \frac{1}{2}(b-a)(\varepsilon_k - 2p^* + 1) \\ &= M_0 + \textstyle\sum_{k=1}^n (1+r)^{-k} \gamma_k S_{k-1}(\rho_k - r) \end{aligned}$$

where γ_k, $k \geq 1$ is a predictable sequence. We set again $X_n = (1+r)^n M_n$ and $\beta_n = (X_{n-1} - \gamma_n S_{n-1})B_{n-1}^{-1}$ so that

$$X_{n-1} = \beta_n B_{n-1} + \gamma_n S_{n-1}.$$

In the same way as in (i) we see that $\pi = (\beta_n, \gamma_n)$, $n \geq 1$ is a self-financing strategy, i.e. that we also have here $X_n = \beta_n B_n + \gamma_n S_n$, $n = 1, 2, ...$ Now, $X_0 = M_0 = Y_0 = V$ and

$$X_n = (1+r)^n M_n = (1+r)^n (Y_n + A_n) \geq (1+r)^n Y_n \geq R_n$$

where the last inequality follows since Y_n is a supremum over all stopping times τ greater or equal n and we can always take $\tau \equiv n$. Hence, π is hedging with the initial capital V and we conlude again that the fair price should not be bigger than V which together with the estimate in the other direction obtained earlier yields that, in fact, it equals V.

(iii) In the game contingent claim case let the payoff function is given by

$$R(m,n) = U_m \mathbb{I}_{m<n} + W_n \mathbb{I}_{m \geq n}$$

where $U_n \geq W_n$. Now, we fix a stopping time σ and define $Z_k^\sigma = B_0 B_{\sigma\wedge k}^{-1} R(\sigma, k)$, $k = 0, 1, ..., N$ and

$$Y_n^\sigma = \max_{n \leq \tau \leq N} E^*(Z_\tau^\sigma | \mathcal{F}_n).$$

It is easy to see (check!) that Z_k^σ is $\mathcal{F}_k$-measurable, and so as we proved it in the optimal stopping section the sequence $\{Y_n^\sigma,\ n = 0, 1, ..., N\}$ is a supermartingale and by the Doob supermartingale decomposition theorem

$$Y_n^\sigma = M_n^\sigma - A_n^\sigma,\ n = 0, 1, ..., N$$

where M_n^σ, $n \geq 0$ is a martingale, $M_0^\sigma = Y_0^\sigma$ and A_n^σ, $n \geq 0$, $A_0^\sigma = 0$ is a non decreasing predictable process.

Again, we use the martingale respresentation lemma for the CRR market to obtain

$$\begin{aligned} M_n^\sigma &= M_0^\sigma + \textstyle\sum_{k=1}^n (1+r)^{-k} \gamma_k^\sigma S_{k-1} \frac{1}{2}(b-a)(\varepsilon_k - 2p^* + 1) \\ &= M_0^\sigma + \textstyle\sum_{k=1}^n (1+r)^{-k} \gamma_k^\sigma S_{k-1} (\rho_k - r) \end{aligned}$$

where γ_k^σ, $k \geq 1$ is a predictable sequence. We set again $X_n^\sigma = (1+r)^n M_n^\sigma$ and $\beta_n^\sigma = (X_{n-1}^\sigma - \gamma_n^\sigma S_{n-1}) B_{n-1}^{-1}$ so that

$$X_{n-1}^\sigma = \beta_n^\sigma B_{n-1} + \gamma_n^\sigma S_{n-1}.$$

In the same way as in (i) we see that $\pi^\sigma = (\beta_n^\sigma, \gamma_n^\sigma)$, $n \geq 1$ is a self-financing strategy, i.e. that we also have here $X_n^\sigma = \beta_n^\sigma B_n + \gamma_n^\sigma S_n$, $n = 1, 2, ...$ Now,

$$X_n^\sigma = (1+r)^n M_n^\sigma = (1+r)^n (Y_n^\sigma + A_n^\sigma) \geq (1+r)^n Y_n^\sigma \geq R(\sigma, n),$$

and so π^σ is a hedging strategy with the initial capital

$$X_0^\sigma = M_0^\sigma = Y_0^\sigma = \max_{0 \leq \tau \leq N} E^*(Z_{\sigma\wedge\tau}).$$

Take $\sigma^* = \min\{n : U_n (1+r)^{-n} = V_n\}$ where

$$V_n = \min_{n \leq \sigma \leq N} \max_{n \leq \tau \leq N} E^*(Z_{\sigma\wedge\tau} | \mathcal{F}_n).$$

It follows from the above formulas and the theorem about Dynkin's games proved before that $X_0^{\sigma^*} = V$, and so π^{σ^*} is a hedging strategy with the initial capital V. Relying on the same way concluding argument as in (i) and (ii) we complete the proof of (iii) and of the whole theorem. □

From the backward induction (dynamical programming) formulas derived for single and game versions of the optimal stopping problems we obtain the following

1.4. **Corollary.** *(i) The fair price V of an American contingent claim can be obtained by the backward induction as $V = V_0$ where $V_N = \frac{B_0 R_N}{B_N}$ and*

$$V_n = \max(B_0 \frac{R_n}{B_n},\, E^*(V_{n+1}|\mathcal{F}_n))$$

for $n = N-1, N-2, ..., 1, 0$;

(ii) The fair price V of a game contingent claim can be obtained by the backward induction as $V = V_0$ where $V_N = \frac{B_0 U_N}{B_N}$ and

$$V_n = \min\big(B_0 \frac{U_n}{B_n},\, \max(B_0 \frac{W_n}{B_n},\, E^*(V_{n+1}|\mathcal{F}_n))\big)$$

for $n = N-1, N-2, ..., 1, 0$ where $R(m,n) = U_m \mathbb{I}_{m<n} + W_n \mathbb{I}_{m \geq n}$.

REFERENCES

[1] Yu. Kifer, *Game options*, Finance Stochast. 4(2000), 443–463.

[2] A.N. Shiryaev, Y.M. Kabanov, D.O. Kramkov, A.B. Melnikov, *To the theory of computations of European and American options I* (Discrete time), Th. Probab. Appl. 39 (1994), 14-60.