AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Lecture 4: General financial market and derivative securities in discrete time

1.1. **General financial market in discrete time.** We start with a probability space together with a filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\geq 0}, P)$ and two types of securities:

a) a bond $B = \{B_n\}_{n\geq 0}$ such that its price $B_n > 0$ at time n is a $\mathcal{F}_{n-1}$-measurable random variable (predictable) and

b) d-stocks $S = (S_n^1, S_n^2, ..., S_n^d)$ where the price $S_n^i > 0$ of ith stock at time n is a $\mathcal{F}_n$-measurable random variable.

We write $B_n = (1 + r_n)B_{n-1}$ and $\Delta B_n = B_n - B_{n-1}$ so that $r_n = \frac{\Delta B_n}{B_{n-1}} > -1$ is $\mathcal{F}_{n-1}$-measurable. Similarly, we have $S_n^i = (1 + \rho_n^i)S_{n-1}^i$, $\Delta S_n^i = S_{n+1}^i - S_n^i$ so that $\rho_n^i = \frac{\Delta S_n^i}{S_{n-1}^i}$ is $\mathcal{F}_n$-measurable. Thus, we have

$$B_n = B_0 \prod_{1\leq k\leq n}(1 + r_k) \quad S_n^i = S_0^i \prod_{1\leq k\leq n}(1 + \rho_k^i).$$

1.1. **Definition.** A pair $\pi = (\beta, \gamma)$ of predictable sequences of random variables $\beta = \{\beta_n\}_{n\geq 0}$ and $\gamma = \{\gamma_n^1, ..., \gamma_n^d\}_{n\geq 0}$ is called a trading strategy (thus β_n, γ_n^i are $\mathcal{F}_{n-1}$-measurable). The value of the corresponding investment portfolio at time n is

$$X_n^\pi = \beta_n B_n + \sum_{i=1}^d \gamma_n^i S_n^i = \beta_n B_n + (\gamma_n, S_n)$$

where $(\cdot, \cdot)$ denotes the inner product. The investment portfolio (and the corresponding trading strategy) is called self-financing if

$$X_{n-1}^\pi = \beta_n B_{n-1} + \sum_{i=1}^d \gamma_n^i S_{n-1}^i$$

or equivalently

$$\Delta X_n^\pi = X_n^\pi - X_{n-1}^\pi = \beta_n \Delta B_n + \sum_{i=1}^d \gamma_n^i \Delta S_n^i = \beta_n \Delta B_n + (\gamma_n, \Delta S_n).$$

We can write

$$\frac{X_n^\pi}{B_n} = \beta_n + \sum_{i=1}^d \gamma_n^i \frac{S_n^i}{B_n}$$

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and

$$\Delta(\frac{X^\pi_n}{B_n}) = \frac{X^\pi_n}{B_n} - \frac{X^\pi_{n-1}}{B_{n-1}} = \sum_{i=1}^d \gamma^i_n(\frac{S^i_n}{B_n} - \frac{S^i_{n-1}}{B_{n-1}})$$
$$= \sum_{i=1}^d \gamma^i_n \Delta\frac{S^i_n}{B_n} = (\gamma_n,\, \Delta\frac{S_n}{B_n}).$$

Summing this equality in n we obtain

$$\frac{X^\pi_n}{B_n} = \frac{X^\pi_0}{B_0} + \sum_{k=1}^n (\gamma_k,\, \Delta\frac{S_k}{B_k}).$$

It follows immediately from the above formulas that

1.2. **Lemma.** *If $\{\frac{S^i_n}{B_n}\}_{n\geq 0}$ is a martingale for each $i = 1, ..., d$ (equivalently $\{\frac{S_n}{B_n}\}_{n\geq 0}$ is a d-dimensional martingale) then $\{\frac{X^\pi_n}{B_n}\}_{n\geq 0}$ is also a martingale.*

The ratios $\frac{S^i_n}{B_n}$ and $\frac{X^\pi_n}{B_n}$ are called adjusted values of the corresponding quantities. Thus, the above lemma sais that if adjusted stock prices form a martingale then adjusted portfolio values (of a self-financing trading strategy) form a martingale, as well. Considering adjusted values usually enables us to study only the case when the bond price is constant (interest zero), i.e. above $r_n \equiv 0$ for all n.

1.2. **Derivative securities.** We will consider three types of derivative securities with a payoff process R.

1.3. **Definition.** (i) A European contingent claim (option or other derivative) is a contract between its buyer and seller so that the latter accepts an obligation to pay an amount $R = R_N$ at time N to the buyer where $R \geq 0$ is a $\mathcal{F}_N$-measurable random variable.

(ii) An American contingent claim (option or other derivative) is a contract between its buyer and seller so that the latter accepts an obligation to pay an amount R_n to the buyer provided the latter exercises the contract at time n. Here $\{R_n\}_{n\geq 0}$ is an adapted to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ nonnegative sequence (i.e. $R_n \geq 0$ is $\mathcal{F}_n$-measurable) and there is a horizon N so that the contract is stopped automatically at time N if it was not exercised before in which case the seller pays to the buyer the amount R_N.

(iii) An Israeli (game) contingent claim (option or other derivative) is a contract between its buyer and seller so that the latter accepts an obligation to pay an amount $R(m,n) = Y_m \mathbb{I}_{m<n} + Z_n \mathbb{I}_{m\geq n}$ provided the seller cancells the contract at time m and the buyer exercises it at time n. Here, $\{Y_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ are nonnegative adapted to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ sequences such that $Y_n \geq Z_n$ a.s. for all n. Again, there is a horizon N so that the contract is stopped automatically at time N if it was not cancelled or exercised before in which case the seller pays to the buyer the amount $R(N,N) = Y_N = Z_N$. The amount $\delta_n = Y_n - Z_n \geq 0$ is interpreted as a penalty the seller has to pay to the buyer for cancellation of the contract.

In order to study the above objects we will always assume that all random variables we deal with are integrable (i.e. expectations with respect to probabilities under consideration of their absolute values are finite).

1.4. **Definition.** (Hedging) A self-financing trading strategy $\pi = (\beta, \gamma)$ is called a hedging strategy or a hedge (and the corresponding portfolio is called a hedging portfolio) in the European and the American contingent claims cases if the portfolio value X^π is kept sufficent a.s. to cover the payment obligation of the seller according to the corresponding contingent claim. This means:

(i) In the European contingent claim case $X_N^\pi \geq R_N$ a.s.;

(ii) In the American contingent claim case $X_n^\pi \geq R_n$ a.s. for all $n = 0, 1, ..., N$;

1.5. **Definition.** An investment strategy of the seller in the game contingent claim case is a pair (π, σ) where $\pi = (\beta, \gamma)$ is a trading strategy and σ is the cancellation of the contract stopping time. Such an investment strategy (π, σ) is called hedging if π is self-financing and $X_{\sigma\wedge n}^\pi \geq R(\sigma, n)$ a.s. for all $n = 0, 1, ..., N$.

1.6. **Definition.** The fair price V of a contingent claim (according to Black-Scholes and Merton) is the infimum of initial capitals of all hedging (self-financing) portfolios, i.e. $V = \inf\{x :$ there exists a hedging trading strategy π (hedging investment strategy (π, σ) in the game options case) such that $X_0^\pi = x\}$.

1.7. **Definition.** A probability measure Q on the probability space $(\Omega, \mathcal{F}, P)$ is called a martingale measure if it is equivalent to P (both measures have the same sets of zero measure), written $Q \sim P$, and the adjusted stock prices $\{\frac{S_n}{B_n}\}_{n\geq 0}$ form a martingale with respect to Q.

1.8. **Theorem.** *(Lower bound of the fair price) Let Q be a martingale measure. Then the fair price V of a contingent claim with a payoff process R satisfies*

(i) in the European case:

$$V \geq B_0 E_Q(\frac{R_N}{B_N}),$$

where E_Q denotes the expectation with respect to Q;

(ii) in the American case:

$$V \geq \sup_{0\leq\tau\leq N} B_0 E_Q(\frac{R_\tau}{B_\tau}),$$

where the supremum is taken over the stopping times;

(iii) in the Israeli (game) case

$$V \geq \inf_{0\leq\sigma\leq N} \sup_{0\leq\tau\leq N} B_0 E_Q(\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}),$$

where both the infimum and the supremum are taken over the stopping times.

Proof. (i) Let π be a hedging trading strategy. Then

$$\frac{X_0^\pi}{B_0} = E_Q(\frac{X_N^\pi}{B_N}) \geq E_Q(\frac{R_N}{B_N})$$

where the equality holds true since $\{\frac{X_n^\pi}{B_n}\}_{n\geq 0}$ is a martingale with respect to Q and the inequality above follows by the definition of hedging, proving (i).

(ii) Let π be a hedging trading strategy. Then for any stopping time $\tau \leq N$ by the optional stopping theorem

$$\frac{X_0^\pi}{B_0} = E_Q(\frac{X_\tau^\pi}{B_\tau}) \geq E_Q(\frac{R_\tau}{B_\tau})$$

where the last inequality holds true by the definition of hedging. This inequality holds true for any stopping time $\tau \leq N$, and so we can take the supremum in the last expression, proving (ii).

(iii) Let (π, σ) be a hedging investment strategy. Then for any stopping time $\tau \leq N$ by the optional stopping theorem

$$\frac{X_0^\pi}{B_0} = E_Q\left(\frac{X^\pi_{\sigma\wedge\tau}}{B_{\sigma\wedge\tau}}\right) \geq E_Q\left(\frac{R(\sigma,\tau)}{B_{\sigma\wedge\tau}}\right).$$

Taking the supremum in τ we prove (iii) (the infimum in σ comes into play only when we deal with the upper bound of the fair price). $\square$

References

[1] A.N. Shiryaev, *Essentials of Stochastic Finance*, World Scientific, Singapore, 1999.

[2] S. Shreve, *Stochastic Calculus for Finance I*, Springer, New York, 2004.