AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Lecture 3: Optimal stopping

1.1. **Single player.**

1.1. **Theorem.** *(see [2]) Let Z_n, $n \geq 0$ be a sequence of random variables adapted to a filtration $\{\mathcal{F}_n,\, n \geq 0\}$ (meaning that Z_n is $\mathcal{F}_n$-measurable for any n) such that $E\sup_n |Z_n| < \infty$. Let N be fixed and denote by Λ_{nN} the set of all stopping times τ such that $N \geq \tau \geq n$. Set*

$$L_n = ess\ sup_{\tau\in\Lambda_{nN}} E(Z_\tau|\mathcal{F}_n).$$

(Here $Q = ess\ sup_\alpha q_\alpha$ is a random variable such that $Q \geq q_\alpha$ a.s. for each α and if $\tilde{Q}$ also satisfies the latter property then $Q \leq \tilde{Q}$ (see [2]). When we will deal with a finite probability space then there will be only finitely many stopping times as above and we can take max *in place of ess sup). Then*

$$L_n = \max(Z_n, E(L_{n+1}|\mathcal{F}_n)) \quad a.s..$$

If the random variables $Z_n, n \geq 0$ are nonnegative then $\{L_n,\, n \geq 0\}$ is the smallest nonnegative supermartingale dominating the sequence $Z_n, n \geq 0$, i.e. satisfying $L_n \geq Z_n$ (Snell's envelope). Furthermore,

$$EL_n = \sup_{\tau\in\Lambda_{nN}} EZ_\tau.$$

Proof. Let $\tau_1, \tau_2 \in \Lambda_{nN}$ and define

$$\tau = \mathbb{I}_{\Omega\setminus\Gamma}\tau_1 + \mathbb{I}_\Gamma\tau_2 \text{ where } \Gamma = \{\omega :\, E(Z_{\tau_1}|\mathcal{F}_n) < E(Z_{\tau_2}|\mathcal{F}_n)\}.$$

Then τ is a stopping time and

$$E(Z_\tau|\mathcal{F}_n) = \mathbb{I}_{\Omega\setminus\Gamma}E(Z_{\tau_1}|\mathcal{F}_n) + \mathbb{I}_\Gamma E(Z_{\tau_2}|\mathcal{F}_n) = \max(E(Z_{\tau_1}|\mathcal{F}_n), E(Z_{\tau_2}|\mathcal{F}_n)).$$

By the property of ess sup (see, for instance, [1]) there exists a sequence $\tau_k \in \Lambda_{nN}$, $\tau_0 \equiv n$ such that $\sup_k E(Z_{\tau_k}|\mathcal{F}_n) = L_n$ a.s. and by the above argument we can choose this sequence so that this supremum is obtained by a monotone non decreasing convergence, i.e.,

$$E(Z_{\tau_k}|\mathcal{F}_n) \uparrow L_n \text{ as } k \uparrow \infty \text{ a.s.}$$

Now

$$L_n \geq E(Z_{\tau_k}|\mathcal{F}_n) \geq E(Z_{\tau_0}|\mathcal{F}_n) = Z_n.$$

Employing the monotone convergence theorem we obtain from above that

$$E(L_n|\mathcal{F}_{n-1}) = \lim_{k\uparrow\infty} \uparrow E(Z_{\tau_k}|\mathcal{F}_{n-1}) \leq L_{n-1}$$

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since $L_{n-1} = \text{ess sup}_{\tau\in\Lambda_{n-1}} E(Z_\tau|\mathcal{F}_{n-1})$ and $\Lambda_{n-1} \supset \Lambda_{nN}$. Since, clearly, $L_{n-1} \geq Z_{n-1}$ we obtain

$$L_{n-1} \geq \max\big(Z_{n-1}, E(L_n|\mathcal{F}_{n-1})\big)$$

and we obtain that L_n, $n \geq 0$ is a supermartingale.

In order to prove that in fact we have equality in the above inequality we have to show the inequality in the other direction. We have

$$Z_\tau = Z_n \mathbb{I}_{\{\tau=n\}} + Z_{\tau\vee(n+1)}\mathbb{I}_{\{\tau>n\}}$$

for $\tau \in \Lambda_{nN}$ and since $\tau \vee (n+1) \in \Lambda_{n+1}$ it follows that

$$E(Z_{\tau\vee(n+1)}|\mathcal{F}_{n+1}) \leq L_{n+1}.$$

Hence, for any $\tau \in \Lambda_{nN}$,

$$\begin{aligned} E(Z_\tau|\mathcal{F}_n) &= Z_n\mathbb{I}_{\{\tau=n\}} + E(Z_{\tau\vee(n+1)}|\mathcal{F}_n)\mathbb{I}_{\{\tau>n\}} \\ &\leq Z_n\mathbb{I}_{\{\tau=n\}} + E(L_{n+1}|\mathcal{F}_n)\mathbb{I}_{\{\tau>n\}} \leq \max(Z_n, E(L_{n+1}|\mathcal{F}_n)), \end{aligned}$$

and so

$$L_n \leq \max(Z_n,\, E(L_{n+1}|\mathcal{F}_n))$$

proving that, in fact, we have here the equality with probability one.

Next,

$$E(\text{ess sup}_{\tau\in\Lambda_{nN}} E(Z_\tau|\mathcal{F}_n)) \geq \sup_{\tau\in\Lambda_{nN}} E(E(Z_\tau|\mathcal{F}_n)) = \sup_{\tau\in\Lambda_{nN}} EZ_\tau.$$

On the other hand,

$$E(\text{ess sup}_{\tau\in\Lambda_{nN}} E(Z_\tau|\mathcal{F}_n)) = \lim_{k\uparrow\infty} \uparrow E(E(Z_{\tau_k}|\mathcal{F}_n)) = \lim_{k\uparrow\infty} \uparrow E(Z_{\tau_k}) \leq \sup_{\tau\in\Lambda_{nN}} E(Z_\tau).$$

Hence,

$$E(\text{ess sup}_{\tau\in\Lambda_{nN}} E(Z_\tau|\mathcal{F}_n)) = \sup_{\tau\in\Lambda_{nN}} E(Z_\tau),$$

i.e.

$$EL_n = \sup_{\tau\in\Lambda_{nN}} E(Z_\tau).$$

If $\tilde{L}_n, n \geq 0$ is another supermartingale which majorates $Z_n, n \geq 0$, i.e. $L_n \geq Z_n$ a.s. then for any $\tau \in \Lambda_{nN}$ by the optional stopping theorem

$$\tilde{L}_n \geq E(\tilde{L}_\tau|\mathcal{F}_n) \geq E(Z_\tau|\mathcal{F}_n),$$

and so $\tilde{L} \geq L_n$. $\square$

Assume that $\mathcal{F}_0$ is the trivial σ-field. If we interpret Z_n as a payoff a player receives upon stopping a game at time n then $L_0 = \sup_{0\leq\tau\leq N} EZ_\tau$ is interpreted as the maximal possible average gain in such a game. We assume that only stopping times can be used since this means that only the information available up to the present n time which is carried by the σ-algebra $\mathcal{F}_n$ is used to decide whether to stop or not (no clairvoyance!). The backward induction (dynamical programming) procedure described in the above theorem allows to compute this maximal gain (game value). The following result describes the optimal stopping time to achieve this gain.

1.2. **Theorem.** *In notations and conditions of the previous theorem define*

$$\tau_0 = \begin{cases} \min\{n : Z_n = L_n\} \text{ if an } n \le N \text{ exists that the event in brackets occurs} \\ N \text{ otherwise} \end{cases}$$

Then $L_0 = EZ_{\tau_0}$.

Proof. On the event $\{\tau_0 > n\}$ we have that $L_n > Z_n$ a.s., and so by the previous theorem on this event $L_n = E(L_{n+1}|\mathcal{F}_n)$. Hence,

$$\begin{aligned} E(L_{\tau_0 \wedge (n+1)}|\mathcal{F}_n) &= L_{\tau_0}\mathbb{I}_{\{\tau_0 \le n\}} + E(L_{n+1}|\mathcal{F}_n)\mathbb{I}_{\{\tau_0 > n\}} \\ &= L_{\tau_0}\mathbb{I}_{\{\tau_0 \le n\}} + L_n\mathbb{I}_{\{\tau_0 > n\}} = L_{\tau_0 \wedge n}. \end{aligned}$$

Hence, the sequence $\{L_{\tau_0 \wedge n},\, n \ge 0\}$ is a martingale, and so $E(L_{\tau_0 \wedge n}) = EL_0$ for all n. In particular, for $n = N$ we have $EL_{\tau_0} = EL_0 = L_0$ since we assumed that $\mathcal{F}_0$ is trivial, and so L_0 is a constant. Since $L_{\tau_0} = Z_{\tau_0}$ we obtain from here that $L_0 = EZ_{\tau_0}$ completing the proof. □

Exercise (Secretary problem). A manager chooses one secretary out of N candidates which arrive at random. The candidates can be ranked in a linear order so that there are no equal candidates. The candidates arrive one in a day in consecutive days and once rejected they do not appear again. The manager wants to maximize probability that the chosen secretary is the best among these candidates. Formulate the corresponding optimal stopping problem, find the optimal stopping time and the corresponding maximal probability to choose the best candidate.

1.2. **Dynkin stopping games.** The modern setup for a Dynkin's game (in discrete time) consists of two players and two $\{\mathcal{F}_n\}$-adapted stochastic processes X_n and Y_n so that when the first player stops the game at time m and the second one stops at time n then the former pays to the latter the amount

$$H(m,n) = X_m\mathbb{I}_{\{m<n\}} + Y_n\mathbb{I}_{\{m \ge n\}}.$$

The time n runs up to some horizon $N < \infty$ when the game is stopped and the first player pays to the second one the amount

$$X_N = Y_N.$$

Next, assume that for any $n \le N$,

$$Y_n \le X_n \quad \mathbb{P} - \text{almost surely (a.s)} \quad \text{and}$$

$$E(|Y_n| + |X_n|) < \infty.$$

It is also possible to consider additional payoff process Z_n so that $Y_n \le Z_n \le X_n$ and the first player pays Z_n if both players stop at the same time but we will not consider this case here. Denote by $\Lambda_{m,n}.\, m \le n$ the collection of all stopping times τ with values between m and n. Introduce the upper and the lower values of the game starting at time $n \le N$ by

$$\bar{V}_n = \text{ess inf}_{\sigma \in \Lambda_{nN}} \text{ess sup}_{\tau \in \Lambda_{nN}} E\big(H(\sigma,\tau)|\mathcal{F}_n\big) \text{ and}$$

$$\underline{V}_n = \text{ess sup}_{\tau \in \Lambda_{nN}} \text{ess inf}_{\sigma \in \Lambda_{nN}} E\big(H(\sigma,\tau)|\mathcal{F}_n\big).$$

1.3. **Theorem.** *(see [2]) Under the above conditions* $V_\tau \overset{def}{=} \bar V_\tau = \underline{V}_\tau$ *a.s. for any stopping time* $\tau \in \Lambda_{0N}$ *and, in particular, the Dynkin's game has a value*

$$V = V_0 = \bar V_0 = \underline{V}_0.$$

Furthermore, the stopping times

$$\sigma_0 = \min\{n \le N : V_n = X_n\} \text{ and } \tau_0 = \min\{n \le N : V_n = Y_n\}$$

are optimal (saddle point), i.e. for any $\sigma, \tau \in \Lambda_{0N}$,

$$E\big(H(\sigma_0,\tau)\big) \le E\big(H(\sigma_0,\tau_0)\big) \le E\big(H(\sigma,\tau_0)\big).$$

We have also the following backward recursive (dynamical programming) relation

$$V_n = \min\big(X_n, \max(Y_n, \mathbb{E}(V_{n+1}|\mathcal{F}_n))\big).$$

Proof. It is clear that $Y_n \le \bar V_n \le X_n$ since the 1st and the 2nd player can always use the stopping times $\sigma \equiv n$ and $\tau \equiv n$, respectively. Let $\sigma \in \Lambda_{nN}$. Set

$$\tilde\sigma = \begin{cases} \sigma & \text{if}\, \sigma > n \\ N & \text{if}\, \sigma = n \end{cases}$$

which is a stopping time (check!). Then

$$\text{ess sup}_{\tau\in\Lambda_{nN}} E(H(\sigma,\tau)|\mathcal{F}_n) = \text{ess sup}_{\tau\in\Lambda_{nN}} E(H(\tilde\sigma,\tau)|\mathcal{F}_n)\mathbb{I}_{\{\sigma>n\}} + X_n\mathbb{I}_{\{\sigma=n\}}.$$

Hence,

$$\begin{aligned}\text{ess sup}_{\tau\in\Lambda_{nN}} E(H(\sigma,\tau)|\mathcal{F}_n) &\ge \min(X_n, \text{ess sup}_{\tau\in\Lambda_{nN}} E(H(\tilde\sigma,\tau)|\mathcal{F}_n) \\ &\ge \min(X_n, E(\bar V_{n+1}|\mathcal{F}_n))\end{aligned}$$

where we used that $\Lambda_{n+1,N} \subset \Lambda_{nN}$ and that

$$E(\bar V_{n+1}|\mathcal{F}_n) \le \text{ess sup}_{\tau\in\Lambda_{n+1,N}} E(H(\sigma,\tau)|\mathcal{F}_n).$$

The latter follows since for any $\sigma \in \Lambda_{n+1,N}$,

$$E(\text{ess sup}_{\tau\in\Lambda_{n+1,N}} E(H(\sigma,\tau)|\mathcal{F}_{n+1})|\mathcal{F}_n) = \text{ess sup}_{\tau\in\Lambda_{n+1,N}} E(H(\sigma,\tau)|\mathcal{F}_n)$$

which is proved similarly to to the single player case by obtaining ess sup as monotone increasing limit and using the monotone convergence theorem, i.e.

$$E(\lim_{k\uparrow\infty} \uparrow E(H(\sigma,\tau_k)|\mathcal{F}_{n+1})|\mathcal{F}_n) = \lim_{k\uparrow\infty} \uparrow E(H(\sigma,\tau_k)|\mathcal{F}_n).$$

Thus,

$$\bar V_n = \text{ess inf}_{\sigma\in\Lambda_{nN}} \text{ess sup}_{\tau\in\Lambda_{nN}} E\big(H(\sigma,\tau)|\mathcal{F}_n\big) \ge \min(X_n, E(\bar V_{n+1}|\mathcal{F}_n)).$$

Next, define

$$\tilde\tau = \begin{cases} \tau & \text{if}\, \tau > n \\ N & \text{if}\, \tau = n \end{cases}$$

which is a stopping time (check!). Then for any $\sigma \in \Lambda_{nN}$,

$$E(H(\sigma,\tau)|\mathcal{F}_n) = E(H(\sigma,\tilde\tau)|\mathcal{F}_n)\mathbb{I}_{\{\tau>n\}} + Y_n\mathbb{I}_{\{\tau=n\}}.$$

Hence for any $\sigma \in \Lambda_{nN}$,

$$\text{ess sup}_{\tau\in\Lambda_{nN}} E(H(\sigma,\tau)|\mathcal{F}_n) \le \max(Y_n, \text{ess sup}_{\tau\in\Lambda_{n+1,N}} E(H(\sigma,\tilde\tau)|\mathcal{F}_n)).$$

Therefore,

$$\begin{aligned}\bar{V}_n &\le \text{ess inf}_{\sigma\in\Lambda_{n+1,N}}\text{ess sup}_{\tau\in\Lambda_{n,N}}E(H(\sigma,\tau)|\mathcal{F}_n)\\ &\le \max(Y_n, \text{ess inf}_{\sigma\in\Lambda_{n+1,N}}\text{ess sup}_{\tau\in\Lambda_{n+1,N}}E(H(\sigma,\tau)|\mathcal{F}_n))\\ &= \max(Y_n, E(\bar{V}_{n+1}|\mathcal{F}_n))\end{aligned}$$

where we used that

$$\begin{aligned}&\text{ess inf}_{\sigma\in\Lambda_{n+1,N}}\text{ess sup}_{\tau\in\Lambda_{n+1,N}}E(H(\sigma,\tau)|\mathcal{F}_n)\\ &= E(\text{ess inf}_{\sigma\in\Lambda_{n+1,N}}\text{ess sup}_{\tau\in\Lambda_{n+1,N}}E(H(\sigma,\tau)|\mathcal{F}_{n+1})|\mathcal{F}_n)\end{aligned}$$

which is proved similarly to above by obtaining ess sup and ess inf as monotone limits and using the monotone convergence theorem.

It follows that

$$\bar{V}_n = \min(X_n, \max(Y_n, E(\bar{V}_{n+1}|\mathcal{F}_n)).$$

In the same way we prove that

$$\underline{V}_n = \min(X_n, \max(Y_n, E(\underline{V}_{n+1}|\mathcal{F}_n)).$$

On the other hand,

$$\bar{V}_N = \underline{V}_N = X_N = Y_N,$$

and so by the backward induction we obtain that $\bar{V}_n = \underline{V}_n$ for all $n = N, N-1, N-2, ..., 1, 0$.

It remains only to show that σ_0 and τ_0 is the optimal (saddle point) pair of stopping times. Indeed, we have from the above that $Y_n \le V_n \le X_n$. Now, on the event $\{\sigma_0 > n\}$ we have that $V_n < X_n$, and so on this event by above

$$V_n = \max(Y_n, E(V_{n+1}|\mathcal{F}_n)) \ge E(V_{n+1}|\mathcal{F}_n).$$

It follows that

$$\begin{aligned}E(V_{\sigma_0\wedge(n+1)}|\mathcal{F}_n) &= V_{\sigma_0}\mathbb{I}_{\{\sigma_0\le n\}} + E(V_{n+1}|\mathcal{F}_n)\mathbb{I}_{\{\sigma_0>n\}}\\ &\le V_{\sigma_0}\mathbb{I}_{\{\sigma_0\le n\}} + V_n\mathbb{I}_{\{\sigma_0>n\}} = V_{\sigma_0\wedge n}.\end{aligned}$$

Hence, the sequence $V_{\sigma_0\wedge n}$ is a supermartingale, and so

$$V_0 \ge EV_{\sigma_0\wedge\tau} \ge EH(\sigma_0,\tau).$$

The latter inequality holds true since on the event $\{\sigma_0 < \tau\}$ we have $V_{\sigma_0\wedge\tau} = V_{\sigma_0} = X_{\sigma_0} = H(\sigma_0,\tau)$ (where the second equality follows by the definition of σ_0) while on the event $\{\sigma_0 \ge \tau\}$ we have $V_{\sigma_0,\tau} = V_\tau \ge Y_\tau = H(\sigma_0,\tau)$ since always $V_n \ge Y_n$.

Similarly, on the event $\{\tau_0 > n\}$ we have that $V_n > Y_n$, and so on this event by above

$$V_n = \min(X_n, E(V_{n+1}|\mathcal{F}_n)) \le E(V_{n+1}|\mathcal{F}_n).$$

It follows that

$$\begin{aligned}E(V_{(n+1)\wedge\tau_0}|\mathcal{F}_n) &= V_{\tau_0}\mathbb{I}_{\{\tau_0\le n\}} + E(V_{n+1}|\mathcal{F}_n)\mathbb{I}_{\{\tau_0>n\}}\\ &\ge V_{\tau_0}\mathbb{I}_{\{\tau_0\le n\}} + V_n\mathbb{I}_{\{\tau_0>n\}} = V_{n\wedge\tau_0}.\end{aligned}$$

Hence, the sequence $V_{n\wedge\tau_0}$ is a submartingale, and so

$$V_0 \le EV_{\sigma\wedge\tau_0} \le EH(\sigma,\tau_0).$$

The latter inequality holds true since on the event $\tau_0 \le \sigma$ we have $V_{\sigma\wedge\tau_0} = V_{\tau_0} = Y_{\tau_0} = H(\sigma,\tau_0)$ while on the event $\{\sigma < \tau_0\}$ we have $V_{\sigma\wedge\tau_0} = V_\sigma \le X_\sigma = H(\sigma,\tau_0)$ since always $V_n \le X_n$.

□

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[2] J. Neveu, *Discrete Parameter Martingales*, North-Holland, Amsterdam/American Elsevier, New York, 1975.