AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Lecture 2: Stopping times and martingale inequalities

1.1. **Optional sampling theorem.**

1.1. **Definition.** Given a filtration $\{\mathcal{F}_t, t \geq 0\}$ (or $\{\mathcal{F}_n, n \in \mathbb{Z}_+\}$) a nonnegative random variable τ is called a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for any t. In discrete time case this is equivalent to say that $\{\tau = n\} \in \mathcal{F}_n$ for any n.

If τ_1 and τ_2 are stopping times then so are $\tau_1 \vee \tau_2 = \max(\tau_1, \tau_2)$ and $\tau_1 \wedge \tau_2 = \min(\tau_1, \tau_2)$.

Example. $\tau_B = \min\{n \geq 0 : X_n \in B\}$ is a stopping time for any Borel set B provided X_n is $\mathcal{F}_n$-measurable for each n.

1.2. **Lemma.** *Let $\{M_n\}_{n\geq 0}$ be a martingale (submartingale) with respect to a filtration $\{\mathcal{F}_n\}$ and τ is a stopping time. Then $M_n^\tau = M_{n\wedge\tau}$ is a martingale (submartingale).*

Proof. We write

$$M_{n\wedge\tau} = \sum_{m=0}^{n-1} M_m \mathbb{I}_{\tau=m} + M_n \mathbb{I}_{\tau\geq n}$$

where $\mathbb{I}_\Gamma = 1$ if an event Γ occurs and $= 0$ for otherwise. Hence,

$$M_{(n+1)\wedge\tau} - M_{n\wedge\tau} = \mathbb{I}_{\tau>n}(M_{n+1} - M_n),$$

and so $E(M_{(n+1)\wedge\tau} - M_{n\wedge\tau}|\mathcal{F}_n) = 0$ in the martingale case or $E(M_{(n+1)\wedge\tau} - M_{n\wedge\tau}|\mathcal{F}_n) \geq 0$ in the submartingale case. □

1.3. **Definition.** If τ is a stopping time and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration. Then $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall \text{t}\}$.

Clearly, τ is $\mathcal{F}_\tau$-measurable.

1.4. **Theorem.** *(Optional sampling theorem) Let $X = (X_n, \mathcal{F}_n)_{n\geq 0}$ (where $X_n, n \geq 0$ is a sequence of random variables and $\{\mathcal{F}_n\}_{n\geq 0}$ is a filtration) be a martingale (submartingale). Let $\tau_1 \leq \tau_2$ be stopping times such that $E|X_{\tau_i}| < \infty$, $i = 1, 2$ and*

$$\liminf_{n\to\infty} \int_{\{\tau_2>n\}} |X_n| dP = 0.$$

Then $E(X_{\tau_2}|\mathcal{F}_{\tau_1}) = X_{\tau_1}$ in the martingale case and $E(X_{\tau_2}|\mathcal{F}_{\tau_1}) \geq X_{\tau_1}$ in the submartingale case.

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Proof. It suffices to prove that for any $A \in \mathcal{F}_{\tau_1}$,

$$\int_A X_{\tau_2} dP = \int_A X_{\tau_1} dP$$

in the martingal case and change = to $\geq$ in the submartingale case. Then it suffices to prove that for any $n \in \mathbb{Z}_+$,

$$\int_{A\cap\{\tau_1=n\}} X_{\tau_2} dP = \int_{A\cap\{\tau_1=n\}} X_n dP$$

in the martingal case and change = to $\geq$ in the submartingale case. Now,

$$\int_{A\cap\{\tau_1=n\}} X_n dP = \int_{A\cap\{\tau_1=n\}\cap\{\tau_2\geq n\}} X_n dP = \int_{A\cap\{\tau_1=n\}\cap\{\tau_2=n\}} X_n dP$$
$$+\int_{A\cap\{\tau_1=n\}\cap\{\tau_2>n\}} X_n dP = (\text{or} \leq \text{in the submartingale case}) \int_{A\cap\{\tau_1=n\}\cap\{\tau_2=n\}} X_{\tau_2} dP$$
$$+\int_{A\cap\{\tau_1=n\}\cap\{\tau_2>n\}} E(X_{n+1}|\mathcal{F}_n) dP = \int_{A\cap\{\tau_1=n\}\cap\{\tau_2=n\}} X_{\tau_2} dP$$
$$+\int_{A\cap\{\tau_1=n\}\cap\{\tau_2\geq n+1\}} X_{n+1} dP = \cdots = (\text{or} \leq \cdots \leq \text{for submartingales})$$
$$\int_{A\cap\{\tau_1=n\}\cap\{n\leq\tau_2\leq m\}} X_{\tau_2} dP + \int_{A\cap\{\tau_1=n\}\cap\{\tau_2>m\}} X_m dP.$$

Letting $m \to \infty$ and using the conditions we derive the result. □

1.5. **Definition.** A sequence $\{X_n\}$ of random variables is uniformly integrable if

$$\lim_{a\to\infty} \sup_n \int_{|X_n|>a} |X_n| dP = 0.$$

1.6. **Corollary.** *If $\{X_n\}$ is a uniformly integrable martingale or submartingale then the conditions of the lemma are satisfied, and so its result holds true, as well. In particular, this is true when $|X_n| \leq C$ for all n and some constant C.*

1.7. **Corollary.** *If $0 \leq \tau_i \leq N$, $i = 1, 2$ and $X_n, n \geq 0$ is a martingale then*

$$EX_0 = EX_{\tau_1} = EX_{\tau_2} = EX_N.$$

1.2. **Martingale ineqalities.**

1.8. **Theorem.** *(see [2]) Let $X_t, t \geq 0$ be a submartingale and $0 \leq t_1 < t_2 < ... < t_n$. (i) Set $X = \max_i X_{t_i}$ and $a > 0$. Then*

$$aP\{X \geq a\} \leq EX_{t_n}\mathbb{I}_{\{X\geq a\}} \leq EX^+_{t_n} \leq E|X_{t_n}|.$$

In particular, if X_t is a martingale and $p \geq 1$ then

$$P\{\max_i |X_{t_i}| \geq a\} \leq \frac{E|X_{t_n}|^p}{a^p}.$$

(ii) Suppose that $X_t \geq 0$ is a submartingale and $p > 1$. Then

$$E(\max_i X^p_{t_i}) \leq (\frac{p}{p-1})^p EX^p_{t_n}.$$

In particular, if X_t is a martingale (not assuming nonnegativity) then

$$E(\max_i |X_{t_i}|^p) \leq (\frac{p}{p-1})^p E|X_{t_n}|^p.$$

Proof. (i) Define the stopping time

$$\tau = \begin{cases} \min\{t_i : X_{t_i} \geq a\} \text{ if an } i \text{ exists that the event in brackets occurs} \\ t_n \text{ otherwise} \end{cases}$$

Then

$$EX_{t_n} \geq EX_\tau = EX_\tau \mathbb{I}_{X \geq a} + EX_{t_n} \mathbb{I}_{X<a} \geq aP\{X \geq a\} + EX_{t_n} \mathbb{I}_{X<a},$$

and so

$$EX_{t_n} \mathbb{I}_{X \geq a} \geq aP\{X \geq a\}$$

implying the assertion (i).

(ii) We have

$$\begin{aligned} EX^p &= \int_\Omega dP \int_0^X p\lambda^{p-1} d\lambda = \int_\Omega dP \int_0^\infty \mathbb{I}_{\{X \geq \lambda\}} p\lambda^{p-1} d\lambda \\ &= p \int_0^\infty \lambda^{p-1} P\{X \geq \lambda\} d\lambda \leq p \int_0^\infty \lambda^{p-2} (EX_{t_n} \mathbb{I}_{\{X \geq \lambda\}}) d\lambda \\ &= p \int_0^\infty \int_\Omega \lambda^{p-2} \mathbb{I}_{\{X \geq \lambda\}} X_{t_n} dP d\lambda = \tfrac{p}{p-1} \int_\Omega X^{p-1} X_{t_n} dP \end{aligned}$$

where we used (i) and integrated

$$\int_0^\infty \lambda^{p-2} \mathbb{I}_{\{X \geq \lambda\}} d\lambda = \int_0^X \lambda^{p-2} d\lambda = \frac{X^{p-1}}{p-1}.$$

By the Hölder inequality $(E|YZ| \leq (E|Y|^p)^{1/p} (E|Y|^q)^{1/q}$, $p, q > 0, 1/p + 1/q = 1)$,

$$EX^{p-1} X_{t_n} \leq (EX^p)^{\frac{p-1}{p}} (EX_{t_n}^p)^{1/p}.$$

It follows that

$$EX^p \leq \frac{p}{p-1} (EX^p)^{\frac{p-1}{p}} (EX_{t_n}^p)^{1/p},$$

and so

$$(\frac{p}{p-1})^p EX_{t_n}^p \geq EX^p$$

completing the proof. □

Remark Another important result about submartingales is the Doob upcrossing inequality which yields the (sub)martingale convergence theorem (see [1]) but we will not need it in this course.

References

[1] P. Billingsley, *Probability and Measure*, 2nd ed., J. Willey, New York, 1986.
[2] A.N. Shiryaev, *Probability*, Springer, New York, 1984.
[3] D. Williams, *Probability with Martingales*, Cambridge Univ. Press, Cambridge, 1991.