AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Introduction

The base setup of Financial Mathematics consists of a financial market with two securities riskless asset (bond, bank account) which evolves deterministically (usually yields some interest) and a risky asset (stock, exchange rates,...) which evolves according to a stochastic process. The price of the riskless asset at time t is denoted B_t and the price of the risky asset is denoted by S_t. Here t runs either along integers $0, 1, 2, ...$ if we consider the discrete time case (and the usually t is replaced by n) or it runs along nonnegative reals $t \geq 0$ if we consider the continuous time case.

An agent acting on the market forms an investment portfolio whose value at time t is $X_t^\pi = \beta_t B_t + \gamma_t S_t$ where $\pi = (\beta_t, \gamma_t)_{t \geq 0}$ is called a portfolio strategy i.e. the way the agent allocates assets between the riskless and the risky ones. A portfolio strategy is called self financing if the agent uses for these allocations only money present in the portfolio and neither infuses new money into portfolio nor takes (spends) money from it for other purposes.

The main object of interest in financial mathematics is a derivative security. A derivative security is a contract between its seller (issuer) and its buyer which obliges the former to pay certain amount to the latter depending on the evolution of the risky asset either at a fixed specified time or at any time by demand from the buyer. There are many different contracts and the most well known ones are put and call options. In the put case the payment fuction is $Y_t = (K - S_t)^+$ and in the call case $Y_y = (S_t - K)^+$ where K is a certain prescribed "strike" price. The meaning is that in the put case the buyer of this option gets the right to sell the risky asset by the price K and if its market price S_t is lower then she/he gets the difference while in the call case the buyer gets the right to buy the risky security for the amount K and if its market price is higher she/he can profit (by selling it in the market).

The original use of derivative securities was a protection against risk. For instance, a company like Volkswagen produces cars in Germany and has expenses in euro while it plans to ship to USA certain amount of cars in, say, 3 months. It sets the price in USD and in order to protect itself from exchange rate fluctuations it buys a corresponding derivative security (which costs money) which ensures that the company will get at least certain exchange rate when it buys back euros for its dollars.

One of the main questions of financial mathematics is the fair pricing of derivative securities. There are different approaches to this question but the most famous is

Date: September 7, 2016.

due to Black-Scholes and Merton (Nobel prize) and it is based on the notion of hedging. A portfolio strategy (and a portfolio itself) is called hedging if its value is sufficient to cover the seller of the derivative security obligation according to the contract. The fair price of the derivative security is the minimal initial amount starting from which it is possible to buid a hedging investment portfolio.

If we disregard in the market such things as transaction costs then the market is called frictionless. Transaction costs and other complications which we will discuss later on make the study more complex and we will disregard them at the beginning. Additional complications arise also depending on which processes we take as describing the evolution of our risky asset. Most of the financial mathematics is based on the machinery of martingales which will be the first topic of our study.

2. Lecture 1: Martingales

2.1. **Conditional expectation.** We always work on a given probability space which is a triple $\Omega, \mathcal{F}, P)$ where Ω is a (sample) space, $\mathcal{F}$ is a σ-algebra and P is a probability measure.

2.1. **Definition.** Let X be a random variable such that $E|X| < \infty$ (finite expectation). Let $\mathcal{A} \subset \mathcal{F}$ be a σ-subalgebra. A random variable Y defined up to probability zero is called the conditional expectation of X with respect to $\mathcal{A}$ and denoted by $Y = E(X|\mathcal{A})$ if for any $A \in \mathcal{A}$,

$$\int_A Y dP = \int_A X dP.$$

Such Y is always exists since if we define $\mu(A) = \int_A X dP$ then this defines a signed measure on $(\Omega, \mathcal{F}, P)$ which is absolutely continuous with respect to P (written $\mu \prec P$), and so by the Radon-Nikodim theorem from Measure Theory we obtain exitence and uniqueness (up to a P-almost sure equality) of Y satisfying the formula above. **Important**: since the conditional expectation is unique only almost surely all equalities and inequalities involving it are understood only almost surely, i.e. up to sets of P-measure zero.

Properties of conditional expectations:

- a) linearity: $E(aX + bY|\mathcal{A}) = aE(X|\mathcal{A}) + bE(Y|\mathcal{A})$, a, b are constants;
- b) if $\mathcal{B} \subset \mathcal{A}$ then $\big(E(X|\mathcal{A})|\mathcal{B}) = E(X|\mathcal{B})$;
- c) if X is $\mathcal{A}$-measurable then $E(X|\mathcal{A}) = X$;
- d) if X is independent of the σ-algebra $\mathcal{A}$ (meaning any events $\{X \leq a\}$ and $A \in \mathcal{A}$ are independent) then $E(X|\mathcal{A}) = EX$;
- e) generalization of c): if X is $\mathcal{A}$-measurable then $E(XY|\mathcal{A}) = XE(Y|\mathcal{A})$;
- f) if $f = f(x, y)$ is a bounded Borel function, X is $\mathcal{A}$-measurable and Y is independent of $\mathcal{A}$ then $E(f(X, Y)|\mathcal{A}) = h(X)$ where $h(x) = Ef(x, Y)$.

2.2. **Lemma. Jensen inequality** *Let φ be a convex function and X is a random variable such that $E|X| < \infty$ and $E|\varphi(X)| < \infty$. Then*

$$E(\varphi(X)|\mathcal{A}) \geq \varphi(E(X|\mathcal{A})).$$

Proof. Since φ is convex then

$$\varphi(x) \geq \varphi(a) + \lambda(a)(x - a),$$

and so

$$\varphi(X) \geq \varphi(E(X|\mathcal{A})) + \lambda(E(X|\mathcal{A}))(X - E(X|\mathcal{A})).$$

Taking the expectation in this inequality we obtain the result. It is legitimate to take the expectation if $E(X|\mathcal{A})$ is bounded. Otherwise, restrict first the inequality to the set (event) $G_n = \{E(X|\mathcal{A}) \leq n\}$ then take the expectation and then let $n \to \infty$. □

Example Let $\Omega = \cup_{i=1}^{\infty} A_i$ and this union is disjoint i.e. $\xi = \{A_i\}_{i=1}^{\infty}$ is a partition of Ω. Let $\mathcal{A}$ be the σ-algebra generated by ξ, i.e. $\mathcal{A}$ is the minimal σ-algebra which contains each $A_i, i = 1, 2, ...$ Let X be a random variable such that $E|X| < \infty$. Then $E(X|\mathcal{A})(\omega) = a_i$ is constant for all (almost all) $\omega \in A_i$. Hence,

$$a_i P(A_i) = \int_{A_i} E(X|\mathcal{A})(\omega) dP = \int_{A_i} X dP,$$

and so $a_i = \frac{1}{P(A_i)} \int_{A_i} X dP$.

2.2. **Martingales, submartingales and supermartingales.**

2.3. **Definition.** Let M_t, $t \in T \subset [0, \infty)$ or $t \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$ be a family of random variables such that $E|M_t| < \infty$ for al t and let $\{\mathcal{F}_t, \, t \in T$ or $t \in \mathbb{Z}_+$ be a filtration of σ-algebras, i.e. a family of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$. Assume that M_t is $\mathcal{F}_t$-measurable for any t. Then the family $\{M_t\}$ is called a martingale with respect to the filtration $\{\mathcal{F}_t\}$ if $E(M_t|\mathcal{F}_s) = M_s$ for all $s < t$. If, instead, we have $E(M_t|\mathcal{F}_s) \geq M_s$ for all $s < t$ ($E(M_t|\mathcal{F}_s) \leq M_s$ for all $s < t$) then the family $\{M_t\}$ is called a submartingale (a supermartingale).

Examples

- 1) Let $X_1, X_2, ...$ be independent identically distributed (i.i.d.) random variables such that $E|X_n| < \infty$ and $EX_n = 0$. Then $M_n = \sum_{j=1}^{n} X_j$ is a martingal with respect to the filtration of σ-algebras $\mathcal{F}_n = \sigma\{X_1, ..., X_n\}$ generated by $X_1, ..., X_n$.
- 2) The sequence $M_m = (X_1 + \cdots + X_n)^2 - nEX_1^2$ is martingale with respect to the same filtration where $X_1, X_2, ...$ are the same as in 1).
- 3) Let $X_1, X_2, ...$ be i.i.d. and such that $X_n > 0$ almost surely (a.s.) and $EX_n = 1$. Then the sequence $M_n = \prod_{j=1}^{n} X_n$ is a martingale with respect to the same filtration as above.
- 4) Let X be an integrable random variable and $\{\mathcal{F}_n, \, n \geq 0\}$ be a filtration. Then $M_n = E(X|\mathcal{F}_n)$, $n = 0, 1, ...$ is a martingale (of Doob).
- 5) If the sequence $\{M_n\}$ is a martingale and φ is a convex function such that $E|\varphi(M_n)| < \infty$ for all n then by Jensen's inequality the sequence $\varphi(M_n)$ is a submartingal. For instance, this is true for $|M_n|^q$ for any $q \geq 1$.

2.4. **Remark.** The name "martingale" probably comes from the strategy of game with the same name when the player doubles her/his bet after each loss. There is also martingale which is the strap used to control hight of horse's head.

2.3. **Supermartingale decomposition.**

2.5. **Theorem.** *Let the sequence $\{Y_n\}_{n\geq 0}$ be supermartingale with respect to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Then there exist a martingale $\{M_n\}_{n\geq 0}$ with respect to the same filtration and a predictable non decreasing process $\{A_n\}_{n\geq 0}$ (predictable means that A_n is $\mathcal{F}_{n-1}$-measurable) such that $A_0 = 0$, $M_0 = Y_0$ and*

$$Y_n = M_n - A_n.$$

The result remains true in the continuous time if the supermartingale and the filtration are right continuous.

Proof. We will consider only the discrete time case which we will mostly need. The proof of the continuous time case is more technical and it is obtained from the discrete time case by an appropriate limiting transition (by subdividing time interval into smaller and smaller pieces). Set $M_0 = Y_0$, $A_0 = 0$ and for $n \geq 1$,

$$M_n = Y_0 + \sum_{k=1}^{n} \big(Y_k - E(Y_k|\mathcal{F}_{k-1})\big) \text{ and } A_n = \sum_{k=1}^{n} \big(Y_{k-1} - E(Y_k|\mathcal{F}_{k-1})\big)$$

and the result follows. □

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