AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Assignment No.5, 11/30/2016, Deadline December 12, 2016

Consider the Black-Scholes market model on a probability space $(\Omega, \mathcal{F}, P)$ where the prices of a stock and of a bond at time t are given by

$$S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t) \text{ and } B_t = B_0 e^{rt},$$

respectively, where W_t is the standard Brownian motion. The market is active on a time interval $[0, T]$, $0 < T < \infty$ and let $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ denotes the Brownian filtration, i.e. $\mathcal{F}_t = \sigma\{W_s,\, s \leq t\}$ is the σ-algebra generated by the random variables $W_s,\, s \leq t$.

We showed in the class that the probability measure $P^{\mu - r}$ defined by means of the Radon-Nikodim derivative

$$\frac{dP^{\mu-r}}{dP} = \exp\big(- \frac{\mu - r}{\sigma} W_T - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 T\big)$$

is a martingale measure, i.e. $\{\frac{S_t}{B_t}\}_{0 \leq t \leq T}$ is a martingale with respect to $P^{\mu-r}$ and the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$.

The fair price of an European contingent claim with a $\mathcal{F}_T$-measurable payoff $f_T \geq 0$ at the time T is given by

$$V = E_{P^{\mu-r}}(e^{-rT} f_T)$$

where $E_{P^{\mu-r}}$ is the expectation with respect to the probability $P^{\mu-r}$.

1) Make the following computations showing all steps of solutions.

(a) Find the explicit expression for the fair price V when the payoff function is given by $f_T = S_T^2$ (square of the stock price at time T), i.e. write and compute explicitly the corresponding integrals.

(b) For the payoff functions $f_T = (S_T - K)^+$, where $K > 0$ is a constant, (call option) and $f_T = (K - S_T)^+$ (put option) represent the corresponding fair price V by means of the normal cumulative distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

2*) (Bonus/honors question). We proved that $P^{\mu-r}$ is a martingale measure. Prove that it is the only (unique) martingale measure.

Date: December 14, 2016.