AN INTRODUCTION TO FINANCIAL MATHEMATICS

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1. Assignment No.1, 9/7/2016

1) Let $X_1, X_2, ...$ be independent random variables such that X_i has the Gaussian distribution with $EX_i = 0$ and $EX_i^2 = \sigma_i^2 > 0$ for all i. Let $\mathcal{F}_n = \sigma\{X_1, X_2, ..., X_n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (i.e. $\mathcal{F}_n$ is generated by $X_1, ..., X_n$ meaning that it is the smallest σ-algebra with respect to which $X_1, ..., X_n$ are measurable). Prove that $M_n = \exp(\sum_{k=1}^n \alpha_k X_k - \frac{1}{2}\sum_{k=1}^n \alpha_k^2 \sigma_k^2)$, $n = 1, 2, ...$; $M_0 = 1$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ where $\alpha_1, \alpha_2, ...$ are arbitrary numbers.

2) Let $X_1, X_2, ...$ be independent random variables such that each X_i has a symmetric distribution (i.e. X_i and $-X_i$ have the same distribution) and assume that $E|X_i|^3 < \infty$ for all i. Prove that $L_n = (\sum_{k=1}^n X_k)^3 - 3(\sum_{k=1}^n X_k)(\sum_{k=1}^n \sigma_k^2)$ is a martingale with respect to the same filtration as in 1).

3) (bonus/honors question) a) What is the connection of 2) to 1)?

b) How to generalize 1) to independent random variables with arbitrary distributions (having moment generating functions) so that a similar exponential expression will be a martingale there?

4) Show that $\tau_1 = \min\{n : \max_{1\le i\le n} X_i \in (a,b)\}$ and $\tau_2 = \min\{n : \min_{1\le i\le n} X_i \in (a,b)\}$ (where $a < b$ are numbers) are stopping times but that $\tau_3 = \min\{n : \max_{n\le i\le 2n} X_i \in (a,b)\}$ is not a stopping time.

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