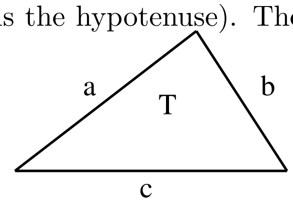
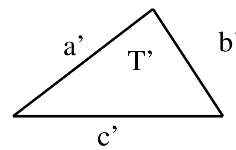
Pythagorean Theorem Using Similar Triangles

Let T be a right triangle whose sides have length a, b, and c (c is the hypotenuse). The Pythagorean Theorem says that



(1) $$a^2 + b^2 = c^2.$$

This is Euclid's second proof (it uses similar triangles). I am surprised that this classical approach that begins by comparing the areas of similar right triangles is not better known.



Let T' with sides a', b', and c' be a right triangle similar to T. Then the corresponding sides of T and T' are proportional, that is, there is a *scaling factor* $t > 0$ so that

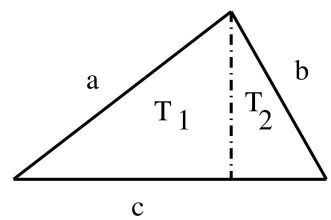
$$a' = ta, \quad b' = tb, \quad c' = tc.$$

First step: Compare Area(T) and Area(T'). Because T and T' are right triangles,

$$\text{Area}(T') = \tfrac{1}{2}a'b' = \tfrac{1}{2}(ta)(tb) = t^2\text{Area}(T).$$

Using $t = \frac{c'}{c}$

(2) $$\text{Area}(T') = \left[\frac{\text{hypotenuse}(T')}{\text{hypotenuse}(T)}\right]^2 \text{Area}(T).$$



Now we use Euclid's key idea: For the right triangle T introduce the altitude to the hypotenuse of T. This partitions T into two right triangles, T_1 and T_2. Both of them are similar to T since their corresponding angles are equal. By comparing the length of the hypotenuse of T, T_1 and T_2 we find the scaling factors:

	T	T_1	T_2
hypotenuse	c	a	b

By equation (2) Area$(T_1) = (a/c)^2$Area(T) and Area$(T_2) = (b/c)^2$Area(T). But Area$(T) =$ Area$(T_1) +$ Area(T_2) so

$$\text{Area}(T) = [(a/c)^2\,\text{Area}(T). + (b/c)^2]\,\text{Area}(T).$$

Dividing by Area(T) gives exactly the Pythagorean formula (1).

Note that he importance of the Pythagorean formula is not for comparing the area of three squares, but for finding the distance between two points in an orthogonal coordinate system.