Applications of Partial Differential Equations To Problems in Geometry

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Preliminary revised version

Preface

These notes are from an intensive one week series of twenty lectures given to a mixed audience of advanced graduate students and more experienced mathematicians in Japan in July, 1983. As a consequence, these they are not aimed at experts, and are frequently quite detailed, especially in Chapter 6 where a variety of standard techniques are presented. My goal was to introduce geometers to some of the techniques of partial differential equations, and to introduce those working in partial differential equations to some fascinating applications containing many unresolved nonlinear problems arising in geometry. My intention is that after reading these notes someone will feel that they can cope with current research articles. In fact, the quite sketchy Chapter 5 and Chapter 6 are merely intended to be advertisements to read the complete details in the literature. When writing something like this, there is the very real danger that the only people who understand anything are those who already know the subject. *Caveat emptor*.

In any case, I hope I have shown that if one assumes a few basic results on Sobolev spaces and elliptic operators, then the basic techniques used in the applications are comprehensible. Of course carrying out the details for any specific problem may be quite complicated—but at least the ideas should be clearly recognizable.

These notes definitely do not represent the whole subject. I did not have time to discuss a number of beautiful applications such as minimal surfaces, harmonic maps, global isometric embeddings (including the Weyl and Minkowski problems as well as Nash's theorem), Yang-Mills fields, the wave equation and spectrum of the Laplacian, and problems on compact manifolds with boundary or complete non-compact manifolds. In addition, these lectures discuss only existence and uniqueness theorems, and ignore other more qualitative problems. Although existence results seem to hold the center of the stage in contemporary applications, a more balanced discussion would be important in a longer series of lectures.

The lectures assumed some acquaintance with either Riemannian geometry or partial differential equations. While mathematicians outside of these areas should be able to follow these notes, it may be more difficult for them to appreciate the significance of the questions or results.

By the ruthless schedule of my charming hosts, these notes are to be typed shortly after the completion of the lectures. My hosts felt (wisely, I think) that it would be more useful to have an informal set of lecture notes available quickly rather than with longer time for a more polished manuscript. Inevitably, as befits a first draft, there will be rough edges and outright errors. I hope none of these are serious and would appreciate any corrections and suggestions for subsequent versions.

One thing I know I would do is add a few additional sections to Chapter

1. In particular, there should really be some mention of Green's functions and at least a vague summary of the story for boundary value problems—especially the Dirichlet problem (see [N-3], pp. 41-50 for what I have in mind). Also, the dry, technical flavor of Chapter 1 should be balanced by a few more easy—but useful—applications of the linear theory. For instance, Moser's result on volume forms [MJ-1] uses only simple Hodge theory. But my time deadline has come.

I hope these notes are useful to someone seeking a rapid introduction with a minimum of background. This task is made much easier because of the recent books [Au-4] and [GT], where one can find most of the missing details. I am grateful to many Japanese mathematicians. In addition to helping make my visit so pleasant, they are also proofreading the typed manuscript; all I'll see is the finished product. Finally, I wish to give special thanks to Professor T. Ochiai for his extraordinary hospitality and thoughtfulness. I also thank the National Science Foundation for their support.

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Note added, June, 1993. This is an essentially unrevised version of the lectures I gave in Japan in July, 1983. The only notable addition is a section discussing the Hodge Theorem, I also took advantage of the retyping into TEX to make a few corrections and minor clarifications in the wording. Alas, retyping introduces its own errors.

[To Do: incorporate the following into the preface]

Throughout these lectures we will need some background material on elliptic and, to a lesser extent, parabolic partial differential operators. Equations that are neither elliptic nor parabolic do arise in geometry (a good example is the equation used by Nash to prove isometric embedding results); however many of the applications involve only elliptic or parabolic equations. For this material I have simply inserted a slightly modified version of an Appendix I wrote for the book [Be-2]. This book may also be consulted for basic formulas in geometry.[2] At some places, I have added supplementary information that will be used later in the lectures. I suggest that one should skim this chapter quickly, paying more attention to the examples than to the generalities, and then move directly to Chapter 6. One can refer back to the introductory material if the need arises.

Most of our treatment is restricted to compact manifolds without boundary. This is simply to avoid the extra steps required to adequately discuss

[2]For reference, some basic geometry formulas are collected in an Appendix at the end of these notes.

appropriate boundary conditions. One can also eliminate most of the complications in thinking about manifolds by restricting attention to the two dimensional torus with its Euclidean metric, so the Laplacian is the basic $u_{xx} + u_{yy}$, and one is considering only doubly periodic functions, say with period 2π. Even simpler, yet still often fruitful and non-trivial, is to reduce to the one dimensional case of functions on the circle. Here $\Delta u = +u''$. This also points out one critical sign convention: for us the Laplacian has the sign so that $\Delta u = +u''$ for functions on $\mathbb{R}^1$ (except that in the special case of the Hodge Laplacian on differential forms, we write $\Delta = dd^* + d^*d$ as in equation (2.4) below, where in the particular case of *0*-forms this gives the *opposite* sign).

To discuss the Laplacian and related elliptic differential operators, one must introduce certain function spaces. It turns out that the spaces one thinks of first, namely C^0, C^1, C^2, etc. are, for better or worse, not appropriate; one is forced to use more complicated spaces. For instance, if $\Delta u = f \in C^k$, one would like to have $u \in C^{k+2}$. With the exception of the special one dimensional case covered by the theory of ordinary differential equations, this is *false* for these C^k spaces (see the example in [Mo, p. 54]), but which is true for the spaces to be introduced now. For proofs and more details see [F, §8-11] and [GT].

Unless stated otherwise, to be safe we will always assume that the open sets we consider are connected.

For simplicity M will always denote a C^∞ *connected Riemannian manifold without boundary*, $n = \dim M$, and E and F are smooth vector bundles (with inner products) over M. Of course, there are related assertions if M has a boundary or if M is not C^∞. Sometimes we will write (M^n, g) if we wish to point out the dimension and the metric, g. The volume element is written dx_g, or sometimes dx. By *smooth* we always mean C^∞; we write C^ω for the space of real analytic functions.

We also use standard multi-index notation, so if $x = (x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$ and $j = (j_1, \ldots, j_n)$ is a vector of non-negative integers, then $|j| = j_1 + \cdots + j_n$, $x^j = x_1^{j_1} \cdots x_n^{j_n}$, and $\partial^j = (\partial/\partial x_1)^{j_1} \cdots (\partial/\partial x_n)^{j_n}$

Here and below we will use the notation $a(x, \partial^k u)$, $F(x, \partial^k u)$, etc. to represent any (possibly nonlinear) differential operator of order k (so here $\partial^k u$ actually represents the k-jet of u).

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Contents

Chapter 1

Linear Differential Operators

1.1 Introduction

Three models from classical physics are the source of most of our knowledge of partial differential equations:

$$\begin{aligned} \text{wave equation:} \quad & u_{xx} + u_{yy} = u_{tt} \\ \text{heat equation:} \quad & u_{xx} + u_{yy} = u_t \\ \text{Laplace equation:} \quad & u_{xx} + u_{yy} = 0. \end{aligned}$$

Because the expression $u_{xx} + u_{yy}$ arises so often, mathematicians generally uses the shorter notation Δu (physicists and engineers often write $\nabla^2 u$).

One thinks of a solution $u(x, y, t)$ of the *wave equation* as describing the motion of a drum head Ω at the point (x, y) at time t. We denote the boundary by $\partial\Omega$. A typical problem is to specify

initial position	$u(x, y, 0)$	
initial velocity	$u_t(x, y, 0)$	
boundary condition	$u(x, y, t)$	for $(x, y) \in \partial\Omega$ and $t \geq 0$.

and seek the solution $u(x, y, t)$. Although we shall essentially not mention the wave equation again in these lectures, it is fundamental.

For the *heat equation*, $u(x, y, t)$ gives the temperature at the point (x, y) at time t. Here a typical problem is to specify

initial temperature	$u(x, y, 0)$	
boundary temperature	$u(x, y, t)$	for $(x, y) \in \partial\Omega$ and $t \geq 0$

and seek $u(x, y, t)$ for $(x, y) \in \Omega$, $t > 0$. This boundary condition is called a *Dirichlet* boundary condition.

As a alternate, instead of specifying the boundary temperature, one might specify that all or part of the boundary in insulated, so heat does not flow across the boundary at those points. Mathematically one writes this as $\partial u/\partial\nu = 0$, where $\partial u/\partial\nu$ means the directional derivative in the direction ν normal to the boundary. This is called a *Neumann* boundary condition. Note that if one investigates heat flow on the surface of a sphere or torus—or any compact manifold *without* boundary—then there are no boundary conditions for the simple reason that there is no boundary.

It is clear that if a solution $u(x, y, t)$ of the heat equation is *independent* of t, so one is in equilibrium, then u is a solution of the *Laplace equation* (it is called a *harmonic* *function*). Using the heat equation model, a typical problem is the *Dirichlet problem*, where one specifies

$$\text{boundary temperature} \quad u(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in \partial\Omega$$

and one seeks the (equilibrium) temperature distribution $u(x,y)$ for $(x,y) \in \Omega$. One might also specify a *Neumann* boundary condition

$$\frac{\partial u}{\partial \nu} = \psi(x,y)$$

on all or part of the boundary.

From these physical models, it is intuitively plausible that in equilibrium, the maximum (and minimum) temperatures cannot occur at an interior point of Ω unless $u \equiv$ const., for if there were a local maximum temperature at an interior point of Ω, then the heat would flow away from that point and contradict the assumed equilibrium. This is the *maximum principle*: if u satisfies the Laplace equation then

$$\min_{\partial\Omega} u \le u(x,y) \le \max_{\partial\Omega} u \qquad \text{for } (x,y) \in \Omega.$$

Of course, one must give a genuine mathematical proof as a check that the model described by the differential equation really does embody the qualitative properties predicted by physical reasoning such as this.

For many mathematicians, a more familiar occurrence of harmonic functions is as the real or imaginary parts of a analytic function $f(z) = u + iv$ of one complex variable z. Indeed, one should expect that harmonic functions have many of the properties of analytic functions. For instance, they will automatically be smooth, and Liouville's theorem holds in the form: "a harmonic function defined on all of $\mathbb{R}^n$ that is bounded below must be a constant." Note that although harmonic functions do form a linear space—since they are the kernel of a linear map—they will not have the additional special algebraic properties of analytic functions: closed under multiplication, inverses $1/f(z)$, and under composition. These algebraic properties of analytic functions are a significant aspect of their special nature and importance.

The *inhomogeneous* Laplace equation $\Delta u = f(x,y)$ is also of importance to us, particularly because in these notes almost all of our discussion will concern compact manifolds without boundary, so there will be no boundary conditions.

In elementary courses in differential equations one main task is to find explicit formulas for solutions of differential equations. This can only be done in the simplest situations, the resulting formulas being fundamental in more advanced work where one must gain insight without such explicit formulas.

EXAMPLE 1.1 [LAPLACE EQUATION ON A TORUS] We will think of the two-dimensional torus T^2 as the square $[0,\, 2\pi] \times [0,\, 2\pi]$ with the sides identified. Thus, smooth functions on the torus will be doubly periodic with period 2π. When can one solve the Laplace equation

$$u_{xx} + u_{yy} = f(x,y)\ ? \tag{1.1}$$

It is natural to use Fourier series. Thus we write f as a Fourier series and seek u as a Fourier series:

$$f(x,y) = \sum f_{k\ell} e^{i(kx+\ell y)}, \qquad u(x,y) = \sum u_{k\ell} e^{i(kx+\ell y)}.$$

The smoothness of these functions will depend on the rate of decay of their Fourier coefficients. Working formally, one substitutes u and f into the differential equation $\Delta u = f$ and matches coefficients

$$\sum -(k^2+\ell^2)u_{k\ell}e^{i(kx+\ell y)} = \sum f_{k\ell}e^{i(kx+\ell y)}.$$

For equality to hold we find that

$$u_{k\ell} = \frac{-f_{k\ell}}{k^2+\ell^2} \tag{1.2}$$

and make the important observation that a necessary condition for a solution to exist is that $f_{00} = 0$, that is, from the formula for the Fourier coefficient

$$\int_{T^2} f(x,y)\,dx\,dy = 0.$$

With hindsight this necessary condition was obvious by just integrating (1.1) over T^2. From our explicit formula for the Fourier coefficients of u, this condition is also sufficient,

$$u(x,y) = \sum \frac{-f_{k\ell}}{k^2+\ell^2}e^{i(kx+\ell y)}$$

Moreover, we see that the Fourier coefficients of u decay more quickly than those of f, so u will be smoother than f. This will be made more precise in Step 6 of Theorem 3.1, where we use Sobolev spaces that will be introduced later in this chapter. After one studies the convergence of the Fourier series, then it is easy to fully justify all of the formal computations we made in this example.

The solution of the Laplace equation is unique, except that one can add a constant to any solution. □

It is useful to remark that the identical approach to solve the wave equation formally on the torus has immediate and serious difficulties because equation (1.2) is replaced by $u_{k\ell} = -f_{k\ell}/(k^2-\ell^2)$, whose denominator is zero whenever $k = \pm\ell$.

EXAMPLE 1.2 [HEAT EQUATION] Let (M^n, g) be a compact Riemannian manifold without boundary; the torus T^2 of the preceding example with the "flat" Riemannian metric $g = dx^2 + dy^2$ is a useful example. We wish to solve the heat equation

$$u_t = \Delta u \qquad \text{for } x \in M, \tag{1.3}$$

where Δ is the *Laplace* (or Laplace-Beltrami) operator of the metric g. The simplest way to define the Laplacian is to require that Green's Theorem holds:

$$\int \nabla u \cdot \nabla\varphi\,dx_g = -\int (\Delta u)\varphi\,dx_g \tag{1.4}$$

for all smooth functions φ with compact support. Here $dx_g = \sqrt{\det g}\,dx = \sqrt{|g|}\,dx$ is the Riemannian element of volume on (M,g).

It is instructive to compute the Laplacian in local coordinates. We use functions φ whose support lies in a coordinate patch. Then writing g^{ij} for the inverse of the metric g_{ij}

$$\nabla u \cdot \nabla\varphi = \sum_{i,j=1}^{n} g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial\varphi}{\partial x^j}$$

so an integration by parts gives

$$\begin{aligned}\int \nabla u\cdot\nabla\varphi\,dx_g &= \int\sum_{i,j} g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial\varphi}{\partial x^j}\sqrt{|g|}\,dx\\ &= -\int\left[\sum_{i,j}\frac{\partial}{\partial x^i}\Big(g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x^j}\Big)\right]\varphi\,dx\\ &= -\int\left[\frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial x^i}\Big(g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x^j}\Big)\right]\varphi\,dx_g. \qquad (1.5)\end{aligned}$$

Comparing the right-hand sides of (1.4) and (1.5) we obtain the desired formula.

$$\Delta u = \frac{1}{\sqrt{|g|}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\Big(g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x^j}\Big). \qquad (1.6)$$

For the flat torus, $g_{ij}=\delta_{ij}$ of course. Our initial condition is

$$u(x,0)=f(x), \qquad (1.7)$$

where f is a prescribed function on M.

Guided by ordinary differential equations we can write the "solution" as

$$u(x,t)=e^{t\Delta}f. \qquad (1.8)$$

To make sense of this we use a spectral representation of Δ. Thus, let λ_j and φ_j be the eigenvalues and corresponding eigenfunctions of $-\Delta$

$$-\Delta\varphi_j=\lambda_j\varphi_j. \qquad (1.9)$$

For the flat torus the eigenvalues are the numbers

$$\lambda_{k\ell}=k^2+\ell^2$$

with corresponding orthonormal eigenfunctions

$$\varphi_{k\ell}=\frac{1}{2\pi}e^{i(kx+\ell y)},$$

where k and ℓ take all possible positive and negative integer values. Although one can compute the eigenfunctions and eigenvalues explicitly for only a few special manifolds, by general theory, it turns out that for any (M,g) the λ_j's, $j=0,1,\dots$ are a discrete set of real numbers converging to ∞. There is a corresponding complete (in $L_2(M)$) set of orthonormal eigenfunctions. Moreover, multiplying (1.9) by φ_j and integrating by parts (or using the divergence theorem if you prefer), we obtain

$$\lambda_j=\frac{\int|\nabla\varphi_j|^2\,dx_g}{\int\varphi_j^2\,dx_g}\ge 0. \qquad (1.10)$$

Here the smallest eigenvalue is $\lambda_0=0$ whose corresponding eigenfunction (normalized to have norm 1) is the constant $\varphi_0=1/\sqrt{\mathrm{Vol}(M)}$.

Formally, we seek a solution of (1.3) as an eigenfunction expansion

$$u(x,t)=\sum a_j(t)\,\varphi_j(x).$$

Substituting this into (1.3) and using the initial condition we obtain

$$u(x,t) = \sum_j f_j e^{-\lambda_j t} \varphi_j(x), \tag{1.11}$$

where

$$f_j = \int f(y)\varphi_j(y)\, dy_g.$$

One can rewrite this solution (1.11) as

$$u(x,t) = \int H(x,y;t) f(y)\, dy_g, \tag{1.12}$$

with

$$H(x,y;t) = \sum_j e^{-\lambda_j t} \varphi_j(x)\varphi_j(y). \tag{1.13}$$

The function H is called *heat kernel* or *Green's function* for the problem (1.3)–(1.7). The formulas (1.12)–(1.13) are our interpretation of (1.8), so $e^{t\Delta}$ is an integral operator (1.12) with kernel H. Then

$$\operatorname{trace} e^{t\Delta} = \int H(y,y;t)\, dy_g = \sum_j e^{-\lambda_j t}. \tag{1.14}$$

We will use this formula in Chapter 2.7.

It is difficult to extract much information from (1.12)-(1.13) unless one has more information on the λ_j 's, φ_j 's or some formula other than (1.13) giving properties of H. These properties depend on the manifold M as well as the metric g. Nontheless, one easy consequence of (1.11) and (1.12) is a simple formula for the equilibrium temperature:

$$\lim_{t\to\infty} u(x,t) = \text{ average of } f = \frac{1}{\mathrm{Vol}(M)} \int f\, dx. \tag{1.15}$$

To prove this, one notes from (1.10) that $\lambda_0 = 0$, $\lambda_j > 0$ for $j \geq 1$ and, as pointed out above, $\varphi_0(x) =$ constant $= \mathrm{Vol}(M)^{-\frac{1}{2}}$. Then by (1.13)

$$\lim_{t\to\infty} H(x,y,t) = \mathrm{Vol}\,(M)^{-1}$$

so the assertion now follows from (1.12). The formula (1.15) states that the equilibrium temperature is the average of the initial temperature—which is amusing but hardly surprising. □

EXAMPLE 1.3 [LAPLACE EQUATION ON A COMPACT MANIFOLD] We can apply the method of the previous example to extend the first example to solve the Laplace equation

$$\Delta u = f$$

on an arbitrary compact connected manifold (M, g) without boundary. As a preliminary step, we observe that the only solution of the homogeneous equation is $u = const.$. This follows by multiplying the equation by u and then integrating by parts:

$$0 = \int_M u\Delta u\, dx_g = -\int_M |\nabla u|^2\, dx_g$$

Thus $\nabla u = 0$ so u is a constant. If we simply integrate $\Delta u = f$ over M, then by the divergence theorem just as on the torus we obtain the necessary condition for solvability

$$0 = \int_M \Delta u \, dx_g = \int_M f \, dx_g$$

To find a formula for a solution we simply replace the use of Fourier series in our discussion of the torus T^2 by the eigenvalues and eigenfunctions of the Laplacian. Thus, we write

$$f(x) = \sum f_k \varphi_k(x) \qquad \text{and we seek} \qquad u(x) = \sum u_k \varphi_k(x),$$

where, since the φ_k are orthonormal, in the L^2 inner product we have

$$f_k = \langle f, \varphi_k \rangle \qquad \text{and} \qquad u_k = \langle u, \varphi_k \rangle.$$

Substituting in the Laplace equation gives

$$u_k = -\frac{f_k}{\lambda_k}.$$

Just as in the case of the torus, because $\lambda_0 = 0$ we again are led to the necessary condition $\langle f, \varphi_0 \rangle = 0$ for solvability. Because φ_0 is a constant, this means that f must be orthogonal to the constants. We can formally write the solution as,

$$u(x) = -\sum \frac{\langle f, \varphi_k \rangle}{\lambda_k} \varphi(x).$$

It is sometimes convenient to rewrite this in the form

$$u(x) = \int_M G(x, y) f(y) \, dy_g \tag{1.16}$$

where we have introduced *Green's function* or *Green's kernel*

$$G(x, y) = -\sum \frac{\varphi_k(x) \varphi_k(y)}{\lambda_k}$$

Conceptually, the advantage of formula (1.16) is that it shows that we should think of this integral operator as the "inverse" of the Laplacian. We must be careful in using the word "inverse" here since there is the necessary condition that f be orthogonal to the constants, and also that the solution u is only unique up to an additive constant. □

Many are dismayed when the solutions of differential equations are presented, as we did in both of our examples, by infinite series. Infinite series are more often thought of as questions than as answers. Yet these infinite series have already yielded some useful information and concepts. They also indicate directions of thought toward proving related results using procedures that do not involve infinite series. The goal of computations is not formulas, it is not numbers. It is insight and understanding. Over the past two centuries the above infinite series have greatly enriched us.

1.2 Hölder Spaces

From calculus one knows that

regularity: if $u'' = f \in C^k$ then $u \in C^{k+2}$
existence: if $f \in C^k$ then there is a $u \in C^{k+2}$ with $u'' = f$.

Thus, one might anticipate that, at least locally,

if $\Delta u = f \in C^k$ then $u \in C^{k+2}$
given any $f \in C^k$ there is some $u \in C^{k+2}$ such that $\Delta u = f$.

Each of these last two assertions is *false* except in dimension one (see the example in [Mo, p. 54]). But they are almost true. The trouble is that the spaces C^k are not really appropriate. After a century we have learned to use the Hölder spaces $C^{k,\alpha}$, where $0 < \alpha < 1$, and Sobolev spaces $H^{p,k}$, $1 < p < \infty$ (here the p is as in the Lebesgue spaces L^p). If in the above assertions one replaces C^k and C^{k+2} by $C^{k,\alpha}$ and $C^{k+2,\alpha}$ (or by $H^{p,k}$ and $H^{p,k+2}$), then they become true.

With this as motivation, we define the Hölder spaces in this section and Sobolev spaces in the next section.

Let $A \subset \mathbb{R}^n$ be the closure of a connected bounded open set and $0 < \alpha < 1$. Then $f : A \to \mathbb{R}$ is *Hölder continuous with exponent* α if the following expression is finite

$$[f]_{\alpha,A} = \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}. \tag{1.17}$$

The simplest example of such a function is $f(x) = |x|^\alpha$ in a bounded set containing the origin. Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set. The *Hölder space* $C^{k,\alpha}(\overline{\Omega})$ is the Banach space of real valued functions f defined on Ω all of whose k^{th} order partial derivatives are Hölder continuous with exponent α. The norm is

$$|f|_{k+\alpha} = \|f\|_{C^k(\overline{\Omega})} + \max_{|j|=k} \ [\partial^j f]_{\alpha,\overline{\Omega}}, \tag{1.18}$$

where $\| \ \|_{C^k(\overline{\Omega})}$ is the usual C^k norm. On a manifold, M, one obtains the space $C^{k,\beta}(M)$ by using a partition of unity. Note that if $0 < \alpha < \beta < 1$, then $C^{k,\beta}(M) \hookrightarrow C^{k,\alpha}(M)$ and by the Arzela—Ascoli theorem, this embedding is compact [For Banach spaces A, B, a continuous map $T : A \to B$ is *compact* if for any bounded set $Q \subset A$, the closure of its image $T(Q)$ is compact. Equivalently, for every bounded sequence $x_j \in A$ there is a subsequence x_{j_k} so that $T(x_{j_k})$ converges to a point in B.]

The Hölder space for $\alpha = 1$ is just the space of Lipschitz continuous functions. They do not (yet) fit into the theory; see [FK] for more recent information.[1]

1.3 Sobolev Spaces

For $f \in C^\infty(M), 1 \le p < \infty$, and an integer $k \ge 0$ define the norm

$$\|f\|_{k,p} = \left[\int_M \sum_{0\le|j|\le k} |D^j f|^p \, dx_g\right]^{1/p}, \tag{1.19}$$

where $|D^j f|$ is the pointwise norm of the j-th covariant derivative. The *Sobolev space* $H^{p,k}(M)$ is the completion of $C^\infty(M)$ in this norm; equivalently, by using local coordinates and a partition of unity, one can describe $H^{p,k}(M)$ as equivalence classes of measurable functions all of whose partial derivatives up to order k are in $L^p(M)$. The space $H^{p,k}(M)$ is a Banach space, and is reflexive if $1 < p < \infty$. If $p = 2$ these spaces

[1] As an exercise, show that if a function is Hölder continuous for some $\alpha > 1$, then it must be a constant.

are Hilbert spaces with the obvious inner products. This simplest case, $p = 2$, is generally adequate for linear problems (such as Hodge theory); nonlinear problems make frequent use of arbitrary values of p. The alternate notation $H^{p,k}$, L^p_k, and W^p_k are often used instead of $H^{p,k}$. For a vector bundle E with an inner product one defines $H^{p,k}(E)$ similarly.[2]

Note that if (within the same differentiable structure) one changes the metric on a compact Riemannian manifold (M, g), then the norms and inner products on the spaces $C^{k,\alpha}(M)$ and $H^{p,k}(M)$ do change; however the new norms are equivalent to the old ones so the topologies do not change.

1.4 Sobolev Embedding Theorem

It is important to investigate relationships among these spaces $C^{k,\alpha}$ and $H^{p,k}$ and also to the familiar spaces $C^k(\Omega)$. For instance, as we shall see shortly, there is a psychologically reassuring fact that if $f \in H^{p,k}$ for all k, then $f \in C^\infty$.

The essence of this study are inequalities relating the various norms. The inequalities are called *Sobolev inequalities.* This is quite simple if Ω is the interval $\Omega = \{0 < x < c\}$ in $\mathbb{R}^1$. For convenience, say $c \geq 1$. Then

$$u(x) = u(y) + \int_y^x u'(t)\,dt \leq |u(y)| + \int_0^c |u'(t)|dt$$

so, integrating this with respect to y we obtain (using $c \geq 1$)

$$|u(x)| \leq \int_0^c (|u'(t)| + |u(t)|)\,dt. \tag{1.20}$$

Using Hölder's inequality for the L^p version, one can rewrite the above as

$$\|u\|_{C^0} \leq \|u\|_{H^{1,1}} \leq c^{1/r}\|u\|_{H^{p,1}} \qquad \text{for} \quad \text{any } p \geq 1, \text{ and } \tfrac{1}{p} + \tfrac{1}{r} = 1.$$

Thus, a Cauchy sequence in $H^{p,1}$ is also Cauchy in C^0, so we have a continuous embedding of $H^{p,1} \hookrightarrow C^0$. Observe that if, say, $u(0) = 0$, then we can let $y = 0$ in the first step above and obtain

$$|u(x)| \leq \int_0^c |u'(t)|\,dt. \tag{1.21}$$

In this case it is particularly clear that the Sobolev inequalities are just generalizations on the mean value theorem, since they show how one can estimate a function in terms of its derivatives. As an exercise, it is interesting to show that (1.21) also holds if one replaces the assumption $u(0) = 0$ with $\int_0^c u = 0$. In general one needs a term involving $|u|$ in (1.20) since otherwise one could add a constant to u and increase the left side but not the right.

In higher dimensions, $\Omega \subset \mathbb{R}^n$, the story is similar but more complicated. The result is called the *Sobolev embedding theorem.*

First we give a few easy but useful observations. One is that if $f \in C^0(M)$ and if we write

$$\|f\|_\infty = \max_{x \in M} |f(x)|$$

[2]While we write $C^\infty(M)$ for smooth real (or complex) valued functions on M, by $C^\infty(E)$ we mean smooth sections of a vector bundle E.

then

$$\lim_{p\to\infty}\|f\|_{L^p(M)} = \|f\|_\infty \tag{1.22}$$

Proof: Given $\epsilon > 0$, let $M_\epsilon = \{x \in M : |f(x)| \geq \|f\|_\infty - \epsilon\}$. Then

$$(\|f\|_\infty - \epsilon)\,\mathrm{Vol}(M_\epsilon)^{1/p} \leq \|f\|_{L^p(M)} \leq \|f\|_\infty\,\mathrm{Vol}(M)^{1/p}.$$

Now let $p \to \infty$.

Another elementary inequality—an immediate consequence of Hölder's inequality applied to $f = 1 \cdot f$ —states that if $1 \leq q \leq p$, then

$$\|f\|_{L^q(M)} \leq \mathrm{Vol}(M)^{(p-q)/pq}\|f\|_{L^p(M)}. \tag{1.23}$$

This shows that if both $\ell \leq k$ and $r \leq p$, then there is a continuous injection $H^{p,k}(M) \hookrightarrow H^{q,\ell}(M)$. We used this above with $k = \ell = 1$ to obtain the L^p version of (1.20) from the L^1 version.The Sobolev Embedding Theorem gives many other such relationships. Among other things, they generalize the mean value theorem in that they give estimates for various norms of a function in terms of norms of its derivatives. Recall that $n = \dim M$ and let $\delta(p,\,k) = k - \frac{n}{p}$.

Theorem 1.1 [Sobolev Inequalities and Embedding Theorem]. *Let* $0 \leq \ell \leq k$ *be integers and assume* $f \in H^{p,k}(M)$.
(a) *If* $\delta(p,k) < \ell$ (that is, $k - \ell < n/p$) *and if* q *satisfies*

$$\delta(q,\ell) \leq \delta(p,k), \qquad \text{equivalently,} \qquad \frac{1}{p} - \frac{k-\ell}{n} \leq \frac{1}{q}, \tag{1.24}$$

then there is a constant $c > 0$ *independent of* f *such that*

$$\|f\|_{\ell,q} \leq c\|f\|_{k,p}. \tag{1.25}$$

Thus there is a continuous inclusion $H^{p,k}(M) \hookrightarrow H^{q,\ell}(M)$. *Moreover, if* $\ell < k$ *and strict inequality holds in* (1.24)*, then this inclusion is a compact operator.*
(b) *If* $\ell < \delta(p,k) < \ell + 1$ (that is, $k - \ell - 1 < n/p < k - \ell$)*, let* $\alpha = \delta(p,k) - \ell$ *so* $0 < \alpha < 1$. *Then there is a constant* c *independent of* f *such that*

$$\|f\|_{\delta(p,k)} = \|f\|_{\ell+\alpha} \leq c\|f\|_{k,p}. \tag{1.26}$$

Thus, there is a continuous inclusion $H^{p,k}(M) \hookrightarrow C^{\delta(p,k)} = C^{k-\frac{n}{p}}(M) = C^{\ell+\alpha}(M)$ *and a compact inclusion* $H^{p,k}(M) \hookrightarrow C^\gamma(M)$ *for* $0 < \gamma < \delta(p,k)$.

For the inclusion $H^{p,k}(M) \hookrightarrow C^{\ell,\,\alpha}$ we naturally identify functions that differ only on sets of measure zero. The compactness assertions of part (a) in this theorem were proved by Rellich for $p = 2$ and generalized by Kondrachov. Note that all of the above results are proved first for a smoothly bounded open set in $\mathbb{R}^n$ and then extended to vector bundles over compact manifolds using a partition of unity.

Some useful special cases (or easy consequences) of the theorem are:

(i) if $f \in H^{p,k}(M)$ and $p > n$, then $f \in C^{k-1}(M)$,

(ii) the inclusion $H^{p,k+1}(M) \hookrightarrow H^{p,k}(M)$ is compact,

(iii) if $f \in H^{p,k}(M)$ and $pk > n$, then $f \in C^0(M)$,

(iv) $C^\infty = \cap_k H^{p,k}$ for any $1 < p < \infty$,

(v) if $f \in H^{2,1}(M)$, then $f \in L^{2n/(n-2)}(M)$ (here $n \geq 3$), and there are constants A, $B > 0$ independent of f such that

$$\|f\|_{L^{2n/(n-2)}} \leq A\|Df\|_{L^2} + B\|f\|_{L^2}. \tag{1.27}$$

The value $q = 2n/(n-2)$ in (1.27) is the largest number for which (1.25) holds in the case $k = 1$, $p = 2$. It is a "limiting case" of the Sobolev inequality. The smallest value of A for which there is some constant B such that (1.27 holds is known (see [GT; p. 151] and also [Au-4]). This smallest constant is independent of the manifold M. On the other hand, for fixed $B > 0$ the smallest permissible value of A does depend on the geometry of M and is related to the isoperimetric inequality (see [Gal]), [SalC]. Related inequalities for limiting cases have been found [T-1], [BW], [Au-2], [L], and play an important role in several recent geometric problems.

Since the condition (1.24) and the related condition in part b) may seem mysterious, it may be useful to point out that they are both optimal and easy to discover by using "dimensional analysis". Because this technique is not as widely known as it should be, we illustrate it for example with $\ell = 0$ in (1.24). Let $\varphi \in C_c^\infty(|x| < 1)$, $\varphi \not\equiv 0$, and let $f_\lambda(x) = \varphi(\lambda x) \in C_c^\infty(|x| < 1)$ for $\lambda \geq 1$. Applying (1.25) with $\ell = 0$ to the f_λ and doing a brief computation, one obtains

$$\|\varphi\|_{L^q} \leq c\lambda^{k+n(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_{k,p}.$$

Letting $\lambda \to \infty$ there is a contradiction unless (1.24) holds. This example uses the conformal map $x \mapsto \lambda x$; it leads one to suspect that conformal deformations of metrics lead one to the limiting case of the Sobolev inequality. This suspicion is verified in Chapter 5.1.

There is a separate collection of related theorems, called *trace theorems,* concerning the restrictions of functions in Sobolev spaces to submanifolds. This is particularly important for boundary value problems since the boundary is usually a submanifold of some sort. A typical result is that if $\Gamma \subset \Omega$ is a smooth hypersurface, then for $k > 1/2$ the restriction operator $\gamma : H^{2,k}(\Omega) \to H^{2,s-\frac{1}{2}}(\Gamma)$ is a continuous map onto all of $H^{2,k-\frac{1}{2}}$. To make sense of this, one needs to define Sobolev spaces $H^{p,k}$ (and related Besov spaces) where k is not necessarily an integer. Since we will not need these results, we forgo further discussion (see [Ad]).

1.5 Adjoint

History sometimes takes a surprising path. Before matrices were even defined the adjoint of a differential operator was introduced by Lagrange (the Lagrange identity for ordinary differential equations); moreover, Green proved the self-adjointness of the Laplacian (Green's second identity). On $\mathbb{R}^1$ with the L^2 inner product, the adjoint of $D = d/dx$ is found simply by integrating by parts: for all φ, $\psi \in C_c^\infty$ (i.e. compact support)

$$\langle \varphi, D\psi \rangle = \int \varphi\bar{\psi}'\,dx = -\int \varphi'\bar{\psi}\,dx = \langle -D\varphi, \psi \rangle.$$

Thus, the adjoint of d/dx is $-d/dx$. More correctly, because d/dx is an unbounded operator on L^2 and thus not defined on the whole Hilbert space, this is the *formal adjoint.* The strict Hilbert space adjoint requires additional attention to the domain of definition of the operator . We used smooth functions with compact support to avoid issues concerning the boundary and smoothness.

The usual rules hold for the adjoint of a sum and the adjoint of a product: $(L+M)^* = L^* + M^*$ and $(LM)^* = M^* L^*$. The second derivative operator D^2 is thus formally self-adjoint.

Similarly, if E and F are smooth Hermitian vector bundles over M and if $P : C^\infty(E) \to C^\infty(F)$ is a *linear* differential operator, then one can use the L^2 inner product to define the *formal adjoint*, P^*, by the usual rule

$$\langle Pu,\, v\rangle_F = \langle u,\, P^* v\rangle_E$$

for all smooth sections $u \in C^\infty(E)_c$, and $v \in C^\infty(F)_c$. Since the supports of u and v can be assumed to be in a coordinate patch, one can compute P^* locally using integration by parts.

EXAMPLE 1.4 If P is the k^{th} order linear differential operator, with possible complex coefficients, then

$$Pu = \sum_{|\alpha|\le k} a^\alpha(x)\, \partial^\alpha u,$$

and

$$P^* v = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \partial^\alpha (\overline{a(x)}_\alpha v).$$

If the coefficients a^α in this example are matrices then, as one should anticipate, the formula for P^* uses the Hermitian adjoint (= conjugate transpose) of the a^α. □

1.6 Principal Symbol

For a linear *constant* coefficient differential operator

$$Pu = \sum_{|\alpha|\le k} a^\alpha \partial^\alpha u,$$

a standard approach to solving $Pu = f$ is to use Fourier analysis. Then, say on $\mathbb{R}^n$, taking the Fourier transform gives

$$P(\xi)\hat{u} = \hat{f}, \tag{1.28}$$

where

$$P(\xi) = \sum_{|\alpha|\le k} i^{|\alpha|} a^\alpha \xi^\alpha$$

is an ordinary polynomial in ξ. To solve the equation one then simply divides both sides of (1.28) by $P(\xi)$ and takes the inverse Fourier transform. We used this method on the first example in Section 1.1 on the torus. As seen already in that example, there could be difficulties because of possible zeroes of $P(\xi)$ and with the convergence of the inverse Fourier transform, but the approach is at least clear in principle. This is essentially how Ehrenpreis and Malgrange, independently, proved that one can always solve $Pu = f$, when $f \in C_c^\infty$.

For the variable coefficient case, one can obtain useful information by freezing the coefficients at one point and examining the corresponding constant coefficient case. This leads one to define the *symbol* of a linear differential operator. To a linear differential operator $P : C^\infty(E) \to C^\infty(F)$ of order k, at every point $x \in M$ and for every

$\xi \in T_x^*M$ one can associate an algebraic object, the *principal symbol* $\sigma_\xi(P;x)$, often written simply $\sigma_\xi(P)$. If, in local coordinates,

$$Pu = \sum_{|\alpha|\le k} a^\alpha(x)\partial^\alpha u, \tag{1.29}$$

where the a^α are $\dim F \times \dim E$ matrices, then $\sigma_\xi(P;x)$ is the matrix

$$\sigma_\xi(P;x) = i^k \sum_{|\alpha|=k} a^\alpha(x)\xi^\alpha \qquad (i=\sqrt{-1}). \tag{1.30}$$

One sometimes deletes the factor i^k here and in (1.31). While this slightly complicates the property (*iii*) below, it eliminates using awkward factors of i in examples in which M could be a real manifold so complex numbers might seem out of place.

To define the principal symbol invariantly, let E_x and F_x be the fibers of E and F at $x \in M$, let $u \in C^\infty(E)$ with $u(x) = z$, and let $\varphi \in C^\infty(M)$ have $\varphi(x) = 0$, $d\varphi(x) = \xi$, then $\sigma_\xi(P;x) : E_x \to F_x$ is the following endomorphism

$$\sigma_\xi(P;x)z = \frac{i^k}{k!}P(\varphi^k u)|_x. \tag{1.31}$$

It is straightforward to verify that this definition does not depend on the choices of either u or φ. This definition shows that the variable ξ in the symbol is an element of the cotangent bundle.

The principal symbol is useful because many of the properties of P depend only on the highest order derivatives appearing in P; the principal symbol is a simple invariant way to refer to this highest order part of P. (It is also sometimes valuable to define the complete symbol, which also includes the lower order derivatives in P, not just its principal part).

To illustrate the value of the principal symbol, shortly we will use an algebraic property of $\sigma_\xi(P)$ to define an elliptic differential operator. This *algebraic* property of ellipticity then will implie *analytic* conclusions, such as the smoothness of solutions of the $Pu = 0$. Before this, we collect several obvious, but useful, algebraic properties;

(i) $\sigma_\xi(P+Q) = \sigma_\xi(P) + \sigma_\xi(Q)$

(ii) $\sigma_\xi(PQ) = \sigma_\xi(P)\sigma_\xi(Q)$

(iii) $\sigma_\xi(P^*) = \sigma_\xi(P)^*$ (Hermitian adjoint of $\sigma_\xi(P)$).

In (*i*) we assume that P and Q have the same order, while in (*ii*) we assume that the composition PQ makes sense. Note that without the factor i^k in (1.30), (1.31), the property (*iii*) would need an extra factor $(-1)^k$ since $(\partial/\partial x)^* = -\partial/\partial x$.

EXAMPLE 1.5 On a manifold M, the exterior derivative, d, acts on the space $\Omega^p(M)$ of smooth differential p-forms, $d : \Omega^p(M) \to \Omega^{p+1}(M)$. It is linear and has the defining property $d(\varphi\alpha) = d\varphi \wedge \alpha + \varphi d\alpha$ for any $\varphi \in C^\infty(M), \alpha \in \Omega^p(M)$. Thus

$$\sigma_\xi(d)\alpha = i\xi \wedge \alpha. \tag{1.32}$$

Similarly, for any vector bundle E over a manifold M, the covariant derivative $D : \Lambda^0(E) \to \Lambda^1(E)$ satisfies $D(\varphi v) = d\varphi \otimes v + \varphi Dv$ for any $\varphi \in C^\infty(M), v \in \Lambda^0(E)$. Consequently

$$\sigma_\xi(D)v = i\xi \otimes v. \tag{1.33}$$

For the heat equation $u_t - \Delta u = 0$, the principal symbol does not contain the time derivative information and is thus a bit too crude for this case.

Chapter 2

Linear Elliptic Operators

2.1 Introduction

If V and W are finite dimensional inner product spaces and $L : V \to W$ is a linear map, one knows that one can solve the equation $Lx = y$ if and only if y is orthogonal to $\ker L^*$. [Proof: we show that $(\text{image}\, L)^\perp = \ker L^*$. Now $z \perp \text{image}\, L \Leftrightarrow \langle Lx,\, z\rangle = 0$ for all $x \Leftrightarrow \langle x,\, L^*z\rangle = 0$ for all $x \Leftrightarrow L^*z = 0$.] This assertion can be summarized by

$$W = L(V) \oplus \ker L^*, \tag{2.1}$$

and can also be formulated as an *alternative*:

> Either one can always solve $Lx = y$, or else $\ker L^* \neq 0$, in which case a solution exists if and only if y is orthogonal to $\ker L^*$.

Application: Let U, V, and W be finite dimensional vector spaces with inner products. If $A : U \to V$ and $B : V \to W$ are linear maps with adjoints A^* and B^*, define the linear map $C : V \to V$ by

$$C = AA^* + B^*B.$$

If $U \xrightarrow{A} V \xrightarrow{B} W$ is *exact* [that is, $\text{image}\,(A) = \ker(B)$], then $C : V \to V$ is invertible. This is a straightforward consequence of (2.1).

One can define the *index* of L by the rule

$$\text{index}\, L = \dim\ker L - \dim\operatorname{coker} L. \tag{2.2}$$

By (2.1) $\dim\operatorname{coker} L = \dim\ker L^*$. Since the matrices L and L^* have the same rank, then $\text{index}\, L = \dim W - \dim V$. It is independent of L and is thus uninteresting. If L is a continuous map between Hilbert spaces, the above reasoning is still valid and shows that $(\text{image}\, L)^\perp = \ker L^*$. Hence $\overline{\text{image}\, L} = (\ker L^*)^\perp$. However, in order to pass to the analog of (2.1) one needs that the image of L be a *closed* subspace; also the index may not be finite.

Fredholm realized that the above alternative, an algebraic property, also holds for linear elliptic differential operators. Moreover, the index is finite—and turns out to be very interesting. In honor of Fredholm, in a Hilbert space we we use the name *Fredholm operator* for one whose image is closed, and whose kernel and cokernel are both finite dimensional; the index is defined for this class of operators.

Solutions of elliptic differential equations, such as the Laplace equation, $u_{xx}+u_{yy} = 0$, also have a striking analytic property that many mathematicians meet first in the special case of the Cauchy-Riemann equations: the solutions are as smooth as possible. For instance, if an elliptic equation has real analytic coefficients, then all solutions are real analytic. As a contrast, solutions of the wave equation $u_{xx} - u_{yy} = 0$, which also has analytic coefficients, need not be smooth since, for instance, any function of the form $u(x, y) = f(x - y)$ is a solution.

2.2 The Definition

A linear differential operator $P : C^\infty(E) \to C^\infty(F)$ is *elliptic at a point* $x \in M$ if the symbol $\sigma_\xi(P; x)$ is an isomorphism for every real non-zero $\xi \in T_x^* M - \{0\}$. It is clear that P being elliptic implies that its formal adjoint, P^*, is also elliptic. Since the definition of the symbol was given invariantly, the definition of elliptic does not depend on a choice of coordinates.

For a system of equations, a necessary condition for ellipticity is that $\dim E_x = \dim F_x$ and that each of the equations in the system have the same order. There is, however, a more general definition of ellipticity for systems, called elliptic in the sense of Douglas-Nirenberg, that allows different orders in the various dependent variables (see [DN] and [ADN-2]).

EXAMPLE 2.1 Consider the second order scalar equation

$$Pu = \sum_{i,j} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j b^j(x) \frac{\partial u}{\partial x_j} + c(x)u, \tag{2.3}$$

where u and the coefficients are real-valued functions. Then for each x and ξ the symbol is the 1×1 matrix

$$\sigma_\xi(P; x) = -\sum_{i,j} a^{ij}(x)\xi_i\xi_j.$$

Hence P is elliptic at x if and only if the matrix $(a^{ij}(x))$ is positive (or negative) definite.

Given a Riemannian metric g, for us the primary example of an elliptic operator is the Laplacian (or Laplace-Beltrami operator) Δ_g (usually written just as Δ) acting on scalar-valued functions. In local coordinates $(x^1, \ldots . x^n)$ with g^{ij} the inverse of g the formula is (1.6)

$$\Delta u := \sum_{i,j=1}^n g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \text{lower order terms.}$$

Ellipticity follows because g_{ij} is positive definite. □

EXAMPLE 2.2 [CAUCHY-RIEMANN] The Cauchy-Riemann equation for a function of one complex variable $z = x + iy$ is

$$\frac{\partial \varphi}{\partial \bar z} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\varphi = F.$$

Its symbol is

$$\sigma_\xi\Big(\frac{\partial}{\partial \bar z}\Big) = \tfrac{i}{2}(\xi_1 + i\xi_2),$$

which clearly shows the ellipticity. The formal adjoint is $(\frac{\partial}{\partial \bar z})^* = \frac{1}{2}(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Note that

$$-4\Big(\frac{\partial}{\partial \bar z}\Big)^*\Big(\frac{\partial}{\partial \bar z}\Big)\varphi = (\varphi_{xx} + \varphi_{yy})$$

is the Laplacian.

Occasionally one splits everything into real and imaginary parts to write the Cauchy-Riemann equations as the usual system of two real equations. Thus, say $\varphi = u + iv$ and $F = a + ib$. Then

$$\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

In this form the symbol is

$$\sigma_\xi(\frac{\partial}{\partial \bar z}) = \tfrac{i}{2}\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1 \end{pmatrix} + \tfrac{i}{2}\begin{pmatrix} 0 & -\xi_2 \\ \xi_2 & 0 \end{pmatrix} = \tfrac{i}{2}\begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1. \end{pmatrix}.$$

It is clearly invertible if $(\xi_1, \xi_2) \neq 0$. □

EXAMPLE 2.3 [HODGE LAPLACIAN] Let $C^\infty(E) \xrightarrow{P} C^\infty(F) \xrightarrow{Q} C^\infty(G)$, where P and Q are first order linear differential operators and E, F, G are Hermitian vector bundles over M. The second order operator

$$L = PP^* + Q^*Q: \ C^\infty(F) \to C^\infty(F) \tag{2.4}$$

is elliptic at x if the following symbol sequence is exact at F_x for every $\xi \in T_x^*M - \{0\}$:

$$E_x \xrightarrow{\sigma_\xi(P;x)} F_x \xrightarrow{\sigma_\xi(Q;x)} G_x \tag{2.5}$$

(The fact that exactness implies ellipticity is a consequence of the APPLICATION after (2.1) above.

The rule (2.4) is a useful construction of an elliptic operator. A particular case is if $P = 0$; in this situation we see that if $\sigma_\xi(Q)$ is injective then Q^*Q is elliptic. An example is where $Q := \nabla$ is the gradient operator. Then $-Q^*$ is the divergence operator and Q^*Q is the Laplacian on functions.

The best-known general instance of this construction is the *Hodge Laplacian* where $E = \Lambda^{p-1}$, $F = \Lambda^p$, and $G = \Lambda^{p+1}$ are spaces of differential forms, and P and Q are both exterior differentiation whose symbol we computed in (1.32). Using this symbol it is easy to verify that the sequence (2.5) is exact (for the exterior algebra, use a basis one of whose elements is ξ). Then the *Hodge Laplacian* $\Delta_H := dd^* + d^*d$ is elliptic. It acts on the space $\Omega^p = C^\infty(\Lambda^p)$ of smooth p-forms. In the special case of $\mathbb{R}^n$ with its standard metric, the Hodge Laplacian on real-valued functions is

$$\Delta_H u = -[u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}].$$

Note the minus sign on the right is the sign convention used by many geometers' and is always used for the Hodge Laplacian, despite inevitable confusion.

□

An operator P is *underdetermined elliptic at* x if $\sigma_\xi(P;x)$ is surjective for every $\xi \in T_x^*M - \{0\}$ (the simplest example is the divergence of a vector field on $\mathbb{R}^n$). In this case PP^* is elliptic at x. Similarly, P is *overdetermined elliptic at* x if $\sigma_\xi(P;x)$ is injective for every $\xi \in T_x^*M - \{0\}$ (the simplest example is the gradient of a real-valued function; another example is the Cauchy-Riemann equation for an analytic function of several complex variables). In this case P^*P is elliptic at x.

2.3 Schauder and L^p Estimates

We observed in the Introduction that to prove a version of the Fredholm alternative for a linear elliptic operator, we need to show that the image is closed. The following functional analysis lemma shows that this is equivalent to proving an inequality.

Lemma 2.1 [PEETRE] *Let X, Y, and Z be reflexive Banach spaces with $X \hookrightarrow Y$ a compact injection and $L : X \to Z$ a continuous linear map. Then the following are equivalent:*
a) The image $L(X)$ is closed and $\ker L$ is finite dimensional,
b) There are constants c_1 and c_2 such that for all $x \in X$

$$\|x\|_X \le c_1\|Lx\|_Z + c_2\|x\|_Y. \tag{2.6}$$

To prove a) $\Rightarrow$ b) write $X = X_1 \oplus \ker L$ so the restriction of L to X_1 is injective. The closed graph theorem then gives (2.6).

To prove b) $\Rightarrow$ a), since $X \hookrightarrow Y$ is compact, the unit ball in $\ker L$ is compact so $\ker L$ is finite dimensional. Now decompose $X = X_1 \oplus \ker L$. Because $L : X_1 \to Z$ is injective and $X \hookrightarrow Y$ is compact, reasoning by contradiction one finds that all $x \in X_1$ satisfy

$$\|x\|_X \le c\|Lx\|_Z \tag{2.7}$$

with some new constant c. Say $Lx_j \to z$ for some $x_j \in X_1$. To show that the image of L is closed we find x in X, so that $z = Lx$. But (2.7) implies the x_j are Cauchy in X so $x_j \to x$ for some $x \in X_1$. Now by continuity $z = \lim Lx_j = Lx$. □

From this lemma, we now understand that the main technical step in the theory of linear elliptic differential operators is establishing an inequality. Recall that E and F are vector bundles over M and that M is compact *without* boundary.[1]

Theorem 2.2 BASIC ELLIPTIC ESTIMATES *Let $P : C^\infty(E) \to C^\infty(F)$ be a linear elliptic differential operator of order k. Then there are constants $c_1, c_2, \ldots, c_6$ such that*

(a) [SCHAUDER ESTIMATES] *for every $u \in C^{k+\ell,\alpha}(E)$,*

$$\|u\|_{k+\ell+\alpha} \le c_1\|Pu\|_{\ell+\alpha} + c_2\|u\|_{C(E)} \le c_3\|u\|_{k+\ell+\alpha} \tag{2.8}$$

(b) [L^p ESTIMATES] *for every $u \in H^{p,k+\ell}(E)$, $1 < p < \infty$,*

$$\|u\|_{p,k+\ell} \le c_4\|Pu\|_{p,\ell} + c_5\|u\|_{L^1} \le c_6\|u\|_{p,k+\ell}. \tag{2.9}$$

Moreover, if one restricts u so that it is orthogonal (in $L^2(E)$) to $\ker P$ then we can let $c_2 = c_5 = 0$ —with new constants c_1 and c_4.

It is conceivable that if we restrict P to $H^{p,k+\ell}(E)$, then $\ker P$ could get smaller as ℓ increases. In fact, since the coefficients in P are smooth, the elliptic regularity Theorem 2.3 shows that $\ker P \subset C^\infty(E)$ so there is no ambiguity. Note that the right-hand sides in (2.8)-(2.9) are obvious. The moral of (2.8)-(2.9) is that in these Hölder and Sobolev spaces, $\|Pu\|$ defines a norm equivalent to the standard norm, except that one must add an extra term if $\ker P \ne 0$, since the $\|Pu\|$ is only a semi-norm. This theorem is proved—in greater generality—in [DN], [ADN-1 and 2], and [Mo, Theorem 6.4.8]; since M has no boundary, all one really needs are the simpler "interior estimates" from these references coupled with a partition of unity argument. (In particular, one does not need the assumption of "proper ellipticity" here, or elsewhere in this chapter.)

[1]In the following we consider a linear differential operator $P : C^\infty(E) \to C^\infty(F)$, of order k; clearly this operator can be extended uniquely to act on $C^{k+\ell,\alpha}(E)$ and $H^{p,k+\ell}(E)$. We presume this extension has been done whenever needed.

While we used Peetre's Lemma 2.1 to motivate the above theorem, in reality the lemma was observed only after the theorem had been found and its usefulness appreciated. The lemma does clarify our understanding why the inequalities in the theorem are basic. Note, too, that it does not apply to the Hölder spaces since they are not reflexive.

In the next two sections these estimates will be used to discuss both the regularity (smoothness) of solutions and the existence of solutions. We'll discuss regularity first, so the these results will be available when we turn to existence.

EXAMPLE 2.4 As a brief preview, for a second order linear elliptic operator P on a smooth manifold M without boundary, we obtain the decomposition

$$L_2(F) = \text{image}\,(P(H^{2,2}) \oplus \ker P^* \tag{2.10}$$

from (2.9) with $p = 2$. Observe that $P: H^{2,2} \to L^2$ is continuous so the proof of (2.2) gives $\ker P^* = (\text{image}\, P(H^{2,2}))^\perp$. Therefore, to prove (2.10) it is enough to show that $\text{image}\, P(H^{2,2})$ is a closed subspace (in any Hilbert space, $(V^\perp)^\perp = \overline{V}$). Because the injection $H^{2,2} \hookrightarrow L^2$ is compact, this follows from Peetre's Lemma 2.1 and the basic inequality (2.9). □

2.4 Regularity (smoothness)

In brief, solutions of elliptic equations are as smooth as the coefficients and data permit them to be. The results are, of course, local. First we consider the case of a linear system,

$$Pu := \sum_{|\alpha| \le k} a^\alpha(x) \partial^\alpha u = f(x). \tag{2.11}$$

Recall that C^ω is the space of real analytic functions.

Theorem 2.3 [REGULARITY] *Assume P is elliptic in an open set $\Omega \subset \mathbb{R}^n$ and that $u \in H^{p,k}(\Omega)$ for some $1 < p < \infty$ satisfies $Pu = f$ (almost everywhere). In the following assume that $\ell \ge 0$ is an integer, $p \le r < \infty$, and $0 < \sigma < 1$.*

a) If $a^\alpha(x) \in C^\ell$ and $f \in H^{r,\ell}$, then $u \in H^{r,k+\ell}$.

b) If $a^\alpha(x) \in C^{\ell,\sigma}$ and $f \in C^{\ell,\sigma}$, then $u \in C^{k+\ell,\sigma}$.

c) If $a^\alpha(x) \in C^\infty$ and $f \in C^\infty$, then $u \in C^\infty$.

d) If $a^\alpha(x) \in C^\omega$ and $f \in C^\omega$, then $u \in C^\omega$.

One can read this theorem as a table, each of the four columns below being separate theorems:

If $a^\alpha(x)$ is in	C^ℓ	$C^{\ell,\sigma}$	C^∞	C^ω
while $f(x)$ is in	$H^{r,\ell}$	$C^{\ell,\sigma}$	C^∞	C^ω
then $u(x)$ is in	$H^{r,k+\ell}$	$C^{k+\ell,\sigma}$	C^∞	C^ω

This theorem is an amalgamation of [ADN II, Th. 10.7], [DN, Th. 4] and [Mo, Theorems 6.2.5 and 6.6.1], where slightly more general results are proved.

Upon first seeing such results, one may wonder if there is any practical situation in which the coefficients are not in C^∞. To answer this effectively, one looks at nonlinear equations, where one often needs results for linear equations with minimal smoothness assumptions (indeed, one even wants results with bounded measurable coefficients). We will see one aspect of this in Example 2.6 below.

EXAMPLE 2.5 To illustrate the use of both the L^p and Schauder theory, we examine a nonlinear equation. Let Ω be a bounded open set, either in $\mathbb{R}^n$ or on a smooth manifold. Say $f(x,s)$ is a bounded C^∞ function on $\Omega \times \mathbb{R}$ and say $u \in H^{2,2}(M)$ is a solution of

$$\Delta u = f(x,u).$$

We claim that, in fact, $u \in C^\infty(\Omega)$. Because $|f(x,u)|$ is bounded, it is in $L^r = H^{r,0}$ for all $r < \infty$. Thus, by the first column of the elliptic regularity theorem 2.3, $u \in H^{r,2}$ for all $r < \infty$. Choosing some $r > n = \dim M$ the Sobolev Embedding Theorem 1.1 then implies that $u \in C^{1,\alpha}$ for some $0 < \alpha < 1$, and therefore so is $\Delta u = f(x,u)$. By the second column of the regularity theorem again $u \in C^{3,\alpha}$. Thus $\Delta u = f(x,u) \in C^{3,\alpha}$, so $u \in C^{5,\alpha}$ etc. This reasoning is often called a "bootstrap argument", since the regularity of u is "raised by its own bootstraps". $\square$

EXAMPLE 2.6 Here is a more complicated instance using a bootstrap argument. Again, let Ω be a bounded open set. Say $u \in C^2$ is a solution of the elliptic equation

$$\sum_{i,j} a^{ij}(x,u,\nabla u)\,\frac{\partial^2 u}{\partial x^i \partial x^j} = f(x,u,\nabla u),$$

where, to insure ellipticity, $a^{ij}(x,s,v)$ is positive definite for all $x \in \Omega$ and all values of the other variables. Both the coefficients $a^{ij}(x,s,v)$ and $f(x,s,v)$ are assumed C^∞ functions of their variables. As in the previous example we will show that $u \in C^\infty$. To prove this we use another bootstrap argument. Since $u \in C^2$ then the functions $a^{ij}(x,u(x),\nabla u(x))$ and $f(x,u(x),\nabla u(x))$ are in C^1 as functions of x, and hence in C^σ for all $0 < \sigma < 1$. By the second column of the elliptic regularity Theorem 2.3 then $u \in C^{2,\sigma}$ for all $0 < \sigma < 1$. Thus a^α and f are in $C^{1,\sigma}$ so $u \in C^{3,\sigma}$ etc. The same proof works if we had assumed only that $u \in H^{p,2}$ for some $p > n$. $\square$

2.5 Existence

The estimates of Theorem 2.2 allow one to prove that for a linear elliptic operator the image of P is closed. As a consequence, the existence theory of a linear elliptic operator on a compact manifold can be stated exactly as in the finite dimensional case stated in the Introduction to this chapter. It is often called the *Fredholm alternative*. Moreover, $\dim\ker P$ is finite so the index, as defined by (2.2) makes sense—and this time it turns out to be interesting since it does depend on the operator.

Theorem 2.4 [FREDHOLM ALTERNATIVE] *Let* $P : C^\infty(E) \to C^\infty(F)$ *be a linear elliptic differential operator of order* k.

(a) *Then both* $\ker P$ *and* $\ker P^* \subset C^\infty$ *and they are also finite dimensional.*

(b) *If* $f \in H^{2,\ell}(F)$, *then there is a solution* $u \in H^{2,k+\ell}(E)$ *of* $Pu = f$ *if and only if* f *is orthogonal in (* $L^2(F)$ *) to* $\ker P^*$; *this solution* u *is unique if one requires that* u *is orthogonal* (in $L^2(E)$) *to* $\ker P$.

(c) *If* $E = F$, *the eigenspaces* $[= \ker(P - \lambda I)]$ *are therefore finite dimensional.*

(d) *Moreover, for* $1 < p < \infty$, *if* $f \in H^{p,\ell}$, $C^{\ell,\alpha}$, *or* C^∞, *then a solution* u *is in* $H^{p,k+\ell}$, $C^{k+\ell,\alpha}$, *or* C^∞, *respectively.*

(e) *For a scalar elliptic operator* $\dim\ker P = \dim\ker P^*$, *so one has "existence if and only if uniqueness".*

The proof for $f \in H^{2,\ell}$ or C^∞ can be found, for example in [W, Chapter 6], while part *d)* is a consequence of the elliptic regularity Theorem 2.3.

For elliptic operators on vector bundles, i.e., for systems of equations, one generalizes part *e)* by using the *index* of P, defined just as for matrices. From part *b)*, $\operatorname{coker} P = \ker P^*$, so, by part *a)* the index is a finite. Part *e)* asserts that a scalar elliptic operator has $\operatorname{index} P = 0$. For the general case of a Fredholm operator L, a critical observation was that $\operatorname{index} L$, which is obviously an integer, does not change if one deforms the operator L continuously. It also does not change if one adds a compact operator to L. This implies that the index of an elliptic operator depends only on topological data and led I.M. Gelfand to suggest that there should be a formula for $\operatorname{index} P$ in terms of topological data of the vector bundle and the symbol of the operator. Atiyah-Singer found that formula. The result has been enormously powerful and useful. Among other things, this formula generalized the Riemann-Roch theorem. In Section 2.7 we will sketch the first step of one approach to proving the Atiyah-Singer index theorem.

It is easy to see that for a linear elliptic operator P, all the information concerning the index is contained in its symbol. If $P : H^{2,m} \to L^2$ has order m, we can write P as $P = P_m + Q$, where P_m involves only derivatives of order m while Q contains all the lower order derivatives. Because $Q : H^{2,m-1} \to L^2$ is continuous and $H^{2,m} \hookrightarrow H^{2,m-1}$ is compact by the Sobolev theorem, we find that $Q : H^{2,m} \hookrightarrow H^{2,m-1} \to L^2$ is a compact perturbation of P. Consequently, the index of P depends only on the highest order terms, so all the information on the index of P is contained in the symbol of P.

The following corollary is in part a restatement of the Fredholm alternative for Hölder and Sobolev spaces. Although we are still assuming the coefficients in our operator are smooth, there are similar versions in more general situations. Here we also extend part of Theorem 2.4 to underdetermined and overdetermined systems. The usefulness of part *b)* below to geometric problems was pointed out in [BE].

Corollary 2.5 *Let* $P : C^\infty(E) \to C^\infty(F)$ *be a linear differential operator of order* k.
(a) *If* P *is either elliptic or underdetermined elliptic, then* $\ker P^* \subset C^\infty$ *is finite dimensional and*

i) $H^{p,\ell}(F) = P(H^{p,k+\ell}(E)) \oplus \ker P^* \quad (1 < p < \infty),$

ii) $C^{\ell,\alpha}(F) = P(C^{k+\ell,\alpha}(E)) \oplus \ker P^*,$

iii) $C^\infty(F) = P(C^\infty(E)) \oplus \ker P^*.$

(b) *If* P *is overdetermined elliptic, then these decompositions remain valid if one replaces* $\ker P^*$ *by the intersection of* $\ker P^*$ *with* $H^{p,\ell}(F)$, $C^{\ell,\alpha}(F)$, *and* $C^\infty(F)$, *respectively (if* $\ell < k$, *then* $\ker P^* \cap H^{p,\ell}(F)$ *are distributions).*

Proof. (a) If P is elliptic, this is immediate. If P is underdetermined elliptic, apply Theorem 2.4 to $Q = PP^*$. Note that since in L^2 $\langle Qv, v\rangle = \langle PP^*v, v\rangle = \|P^*v\|^2$ then $\ker Q = \ker P^*$.

(b) First we prove the portion using part (a)(i) of Corollary 2.5. Since P^*P is elliptic and—in the $L^2(F)$ inner product— $\operatorname{image} P^*$ is orthogonal to $\ker P$ $(= \ker P^*P)$, by (i) of Corollary 2.5 for any $f \in H^{p,\ell}(F)$ there is a solution $u \in H^{p,\ell+k}(E)$ of $P^*Pu = P^*f$. Thus $Pu - f \in \ker P^* \cap H^{p,\ell}(F)$. But Pu is orthogonal to $\ker P^* \cap H^{p,\ell}(F)$, since if $\Psi \in (\ker P^*) \cap H^{p,\ell}(F)$ then in L^2, $\langle Pu, \Psi\rangle = \langle u, P^*\Psi\rangle = 0$. Therefore $Pu - f = 0$ if and only if f is orthogonal to $\ker P^* \cap H^{p,\ell}(F)$. The proof of the remaining cases where $f \in C^{k,\alpha}$ or C^∞ is similar. □

EXAMPLE 2.7 The existence result is even interesting for ordinary differential equations, although it is rarely mentioned. We work on the circle S^1, which is the simplest compact manifold without boundary.

Let $Lu = u' + a(x)u$ for x on the circle S^1 and $a \in C^\infty(S^1)$ a real valued function. Then $L^*v = -v' + a(x)v$, $\dim\ker L^* \le 1$ and one can solve $u' + a(x)u = f(x) \in C^\infty(S^1)$ if and only if $\int_{S^1} f(x)z(x)\,dx = 0$ for all $z \in \ker L^*$.

The special case where $a(x) \equiv 0$ so $z(x) \equiv 1$ is especially easy. For the general case, we obtain the standard explicit formula—in detail—since it is all too often viewed as a complicated formula without any insight that it illustrates an important basic idea in an elementary setting. Just as with diagonalizing matrices, one seeks a change of variable $u = qw$ that simplifies the problem (this is probably the simplest example of a "gauge transformation"). Here q is an non-zero function but for a system of equations, where u is a vector and a a matrix, q is an invertible matrix. Then $Lu = qw' + (q' + aq)w$. This clearly simplifies if we choose q so that $q' + aq = 0$. Then the equation for w is thus $qw' = f$. Formally, if we let $D = d/dx$, then we can write this symbolically as $qD(q^{-1}u) = f$ so the operator $L = qDq^{-1}$ is "similar" to the simple operator D. Thus

$$w = \int q^{-1} f \qquad \text{so} \qquad u(x) = cq(x) + q(x)\int_0^x q^{-1}(t)f(t)\,dt,$$

where c is a constant (again note that formally, $L^{-1} = qD^{-1}q^{-1}$, as expected). All this is local. Since we want u to be a smooth function on the circle, then we need u to be periodic:

$$0 = u(1) - u(0) = cq(1) + q(1)\int_0^1 q^{-1}(t)f(t)\,dt - cq(0). \tag{2.12}$$

If $q(1) - q(0) \ne 0$, this can be solved uniquely for c. If $q(1) - q(0) = 0$, then the kernel of L (on functions on the circle) is not zero and (2.12) becomes a condition for the solvability. Since q^{-1} is a solution of the homogeneous adjoint equation, (2.12) is the condition we sought.

You may find it interesting to extend this to the case of a first order linear system on the circle. Then pick $q(x)$ to be a matrix solution of $q' + aq = 0$ with $q(0) = I$. You may find it helpful to observe that q^{*-1} is a solution of the homogeneous adjoint equation $L^*v = -v' + a^*v = 0$. □

EXAMPLE 2.8 It is easy to prove directly that the elementary one dimensional equation $u'' = f$ on the circle, S^1, has a solution if and only if f is orthogonal to the constants, that is,

$$\int_{S^1} f(x)\,dx = 0.$$

Note that in this case, the constants are in the kernel of the homogeneous equation (the local solution $u(x) = cx$ is a global smooth function on S^1 if and only if $c = 0$). Another useful exercise is to analyze the solvability of $u'' + u = f$ on S^1, where to be specific, we fix that S^1 is the circle $0 \le x \le 2\pi$ with the end points identified. □

EXAMPLE 2.9 In order to apply the existence portion of these results and solve $Pu = f$ on M, one needs to know that $\ker P^* = 0$. As an example of a case that arises frequently, consider the scalar equation

$$Pu = -\Delta u + c(x)u, \tag{2.13}$$

where $c(x) > 0$ (recall the sign convention $\Delta u = +u''$ on $\mathbb{R}$). We present two proofs that $\ker P = 0$. The first uses the obvious *maximum principle* that if $Pu \ge 0$ then $u \le 0$, that is, u can not have a positive local maximum—since at such a point $-\Delta u \ge 0$ and $cu > 0$ so $Pu > 0$ there. If $u \in \ker P$, then it can not have a positive maximum or negative minimum. Hence $u = 0$. (In Section 2.6 we will prove the stronger version that assumes only $c \geq 0$). This basic idea is equally applicable to real second order nonlinear scalar equations.

For the second proof, multiply the equation $Pu = 0$ by u and integrate over M, then integrate by parts (the divergence theorem) to obtain

$$0 = \int_M u(-\Delta u + cu)\, dx_g = \int_M (|\nabla u|^2 + cu^2)\, dx_g.$$

Since $c > 0$, then clearly $u = 0$. The same proof still works if $c \geq 0\,(\not\equiv 0)$. Moreover, it is also applicable to vector-valued functions u with c a positive definite matrix—and similar equations on vector bundles. Bochner and others have used it effectively to prove "vanishing theorems" in geometry. See Chapter 3 below.

In this example, $P = P^*$. Thus $\ker P^* = 0$; we conclude that for any $f \in C^\infty(M)$ there is a unique solution of $-\Delta u + cu = f$. Moreover, $u \in C^\infty$.

If $c(x) \equiv 0$, both of these methods of proof show that on scalar functions $\ker \Delta$ is the constant function. Consequently:

$$\text{One can solve} \quad \Delta u = f \quad \text{if and only if} \quad \int_M f\, dx_g = 0. \quad \square \tag{2.14}$$

In some respects our definition of ellipticity is more general than one might suspect — or desire. Here is an example, due to R.T. Seeley [S], of an elliptic operator P having *every* complex number λ as an eigenvalue. Let M be any compact Riemannian manifold with Laplacian Δ and let $0 \leq \theta < 2\pi$ be a coordinate on the circle S^1. Then the operator

$$Pu = -\Big(e^{-i\theta}\frac{\partial}{\partial\theta}\Big)^2 u - e^{-2i\theta}\Delta u \tag{2.15}$$

(or take the real and imaginary parts if one prefers a pair of real equations) is elliptic on $S^1 \times M$. Making the change of variable $t = \sqrt{\lambda}e^{i\theta}$, we see that for any complex $\lambda \neq 0$, $u = \exp(\pm i\sqrt{\lambda}\exp i\theta)$ is an eigenfunction with eigenvalue λ, while if $\lambda = 0$ then $u \equiv 1$ is an eigenfunction.

This awkward situation does not occur for "strongly elliptic operators". To define these let $P : C^\infty(E) \to C^\infty(E)$ and regard the symbol in local coordinates as a square matrix whose ij element is $[\sigma_\xi(P;x)]_{ij}$. *Strong ellipticity at* x means that for some some $c > 0$ (γ and c may depend on x, but not on ξ or η) the following quadratic form $Q(\eta)$ is definite:

$$Q(\eta) = \mathrm{Re}\Big\{\sum_{i,j}[\sigma_\xi(P;x)]_{ij}\eta_i\bar{\eta}_j\Big\} \geq c|\eta|^2 \tag{2.16}$$

for all complex vectors η and all real vectors $\xi \in T_x^*M$ with $|\xi| = 1$. Replacing ξ by $-\xi$ reveals that the order of P must be even. The Hodge Laplacian on differential forms or tensors is strongly elliptic, for example, while the Cauchy-Riemann equation on $\mathbb{R}^2$, equation (2.15), and the second order operator $(\partial/\partial x + i\,\partial/\partial y)^2$ on $\mathbb{R}^2$ are not strongly elliptic.

Theorem 2.6 [Mo, 6.5.4] *If* $P : C^\infty(E) \to C^\infty(E)$ *is strongly elliptic, then it is elliptic (clearly) and its eigenvalues are discrete, having a limit point only at infinity.*

2.6 The Maximum Principle

The maximum principle is a standard tool for second order scalar elliptic equations. The essential idea has already been used in Example 2.9. Its strength, both for linear and especially nonlinear equations is that it makes only modest assumptions and the technique applies in a variety of situations. Despite its elementary character, by ingenious arguments one can apply the maximum principle to prove deep results. One instance is Alexandrov's moving plane technique which he first used to study embedded hypersurfaces with constant mean curvature;another is Caffarelli's proof of Krylov's basic boundary estimate (see [K-3, Chapter IV.3]).

We will prove several different versions of the maximum principle. All of them begin with the following weak maximum principle.

In local coordinates in a connected open set Ω in either $\mathbb{R}^n$ or an $n-$ dimensional manifold, consider the scalar elliptic operator

$$Lu = -\sum_{i,j} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j b^j(x)\frac{\partial u}{\partial x_j}, \tag{2.17}$$

where we assume the *uniform ellipticity* condition for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$

$$\mu|\xi|^2 \leq \sum a^{ij}\xi_i\xi_j \leq m|\xi|^2$$

for some $\mu, m > 0$. Also assume the coefficients $b = (b^1, \ldots, b^n)$ are bounded, $|b| <$ const.. It will be clear in our proofs that we will usually only need that these assumptions hold locally, in the neighborhood of each point.

Theorem 2.7 [WEAK MAXIMUM PRINCIPLE]
a) *If* $u \in C^2(\Omega)$ *satisfies* $Lu < 0$ *in* Ω, *then* u *cannot attain a local maximum.*
b) *If* $u \in C^2(\Omega)$ *satisfies* $Lu + c(x)u < 0$ *in* Ω *and* $c(x) \geq 0$, *then* u *cannot attain a local non-negative maximum.*

Proof. If u had a local maximum at p, then the first derivatives of u would be zero there and the hessian matrix $u_{ij} = \partial^2 u/\partial x^i \partial x^j$ would be negative semidefinite at p. Thus $\sum a^{ij}(p)u_{ij}(p) \leq 0$ (indeed, if $A = (a_{ij})$ is a positive definite matrix and $B = (b_{ij})$ is negative semidefinite, then $\sum a_{ij}b_{ij} \leq 0$, as one can see more easily by diagonalizing A). Consequently $Lu \geq 0$ at p, which contradicts $Lu < 0$. The proof of part b) is identical.
□

Replacing u by $-u$ one immediately obtains a corresponding minimum principle. This is true throughout this section.

Motivated by the classical version of maximum principle that holds for the Laplace equation $u_{xx} + u_{yy} = 0$, one may ask if there is a version of the maximum principle of part a above that assumes only $Lu \leq 0$. The result is the strong maximum principle, due to E. Hopf. It is proved by a technical device which reduces the proof to the weak maximum principle. One can organize the proof in several different ways. We will begin with a boundary point maximum principle which is important by itself.

Theorem 2.8 [BOUNDARY POINT MAXIMUM PRINCIPLE]
a) *Assume that* $u \in C^2(\Omega)$, $u < M$ *satisfies* $Lu \leq 0$ *in* Ω, *and that* $u(p) = M$ *at a point* $p \in \partial\Omega$. *Also assume that* p *is on the boundary of a ball* B *whose closure is in* $\Omega \cup \{p\}$. *If the outer normal directional derivative,* $\partial u/\partial \nu$, *exists at* p, *then either*

$$\left.\frac{\partial u}{\partial \nu}\right|_p > 0 \qquad \text{or} \qquad u \equiv \text{constant}. \tag{2.18}$$

b) *Instead, assume* u *satisfies* $Lu + cu \leq 0$ *where* $c(x)$ *is a bounded function. If either* $M \geq 0$ *and* $c(x) \geq 0$, *or if* $M = 0$ *with no sign condition on* $c(x)$, *then* (2.18) *still holds.*

Proof. In order to treat both parts of the theorem together, we will think of part *a)* as the case where $c \equiv 0$ and let $c^+(x) = \max(c(x), 0)$.

Let R be the radius of the ball B and r be the distance from its center, which we may assume is at the origin. We claim there is a smooth function $v = v(r)$ with the properties *i)* $v > 0$ in B, *ii)* $v = 0$ and $v_r < 0$ on the boundary of B, and *iii)* in the annular region $A = \{r : R/2 < r < R\} \subset B$ we have $Lv + c^+v < 0$.

Before we exhibit v, we show how to use it to prove the theorem. Let $w = u - M + \epsilon v$. Since $u(x) < M$ on $|x| = R/2$, we can choose $\epsilon > 0$ so that $w(x) \le 0$ on $|x| = R/2$. Together with $v(R) = 0$, this gives $w(x) \le 0$ on the whole boundary of A. Observe that for either parts *a*), where $c \equiv 0$, or *b*), because $(c - c^+)(u - M) \ge 0$, we know that

$$(L + c^+)(u - M) = (L + c)(u - M) - (c - c^+)(u - M) \le -cM \le 0.$$

Therefore $Lw + c^+w < 0$. Thus by the weak maximum principle $w \le 0$ throughout A. Because $w(p) = 0$ this implies that its outer normal derivative (if it exists) satisfies $\partial w/\partial\nu|_p \ge 0$, that is,

$$\left.\frac{\partial u}{\partial \nu}\right|_p \ge -\epsilon \left.\frac{dv}{dr}\right|_p > 0.$$

We exhibit v. Let $v(r) = e^{-\lambda r^2} - e^{-\lambda R^2}$, where $\lambda > 0$ will be chosen shortly. Clearly $v > 0$ in B and both $v = 0,\ v_r < 0$ on ∂B. Also, using the uniform ellipticity, bounds on the coefficients, and that $R/2 < r < R$ in A, we get

$$\begin{aligned}(L + c^+)v &= e^{-\lambda r^2}[-4\lambda^2 \sum_{ij} a^{ij}x_ix_j + 2\lambda \sum_i (a^{ii} + b^i x_i)] + c^+v \\ &\le e^{-\lambda r^2}[-4\lambda^2 \mu R^2 + 2\lambda(nm + |b|R) + c^+].\end{aligned}$$

Therefore, by choosing λ sufficiently large we can insure that $Lv + c^+v < 0$ in A. □

The strong maximum principle is a consequence of this boundary point maximum principle.

Theorem 2.9 [STRONG MAXIMUM PRINCIPLE]
a) *Assume* $u \in C^2(\Omega)$ *satisfies* $Lu \le 0$ *in* Ω. *If* u *has a local maximum, then it is constant.*
b) *Assume* $u \in C^2(\Omega)$ *satisfies* $Lu + c(x)u \le 0$ *in* Ω, *where* $c(x) \ge 0$. *If* u *has a local non-negative maximum, then it is constant. Moreover, if* $c(x) > 0$ *somewhere, then to satisfy* $Lu + c(x)u \le 0$ *this constant must be zero.*
c) *Assume* $u \in C^2(\Omega)$ *satisfies* $Lu + c(x)u \le 0$ *in* Ω *and* $u(x) \le 0$. *Then either* $u < 0$ *or* $u \equiv 0$ (*there is* no *assumption on* c).

Proof. We treat parts *a*) and *b*) together. Since the connectedness of Ω is essential, we will use it explicitly. Say u has a local maximum M at some interior point of Ω. Let $\Omega_M \subset M$ be the set where $u(x) = M$. It is evident that this set is closed and, by assumption, not empty. We show that it is open. Because Ω is connected, this will prove that Ω_M, is all of Ω and thus that u is constant.

To show that Ω_M is open, say $u(p) = M$. Pick a sufficiently small δ so that the ball $|x - p| < 2\delta$ is in Ω. We claim that $u \equiv M$ in the smaller ball $|x - p| < \delta$. This will show that Ω_M is open.

Reasoning by contradiction, assume $u(x_0) < M$ for some x_0 in the smaller ball. Pick the largest ball B centered at x_0 so that $u < M$ in B but $u(q) = M$ for at least one point q on ∂B, the boundary of B (since $u(p) = M$, the radius of B is at most δ). We can now apply the boundary point maximum principle, Theorem 2.8, to find a directional derivative at q that is positive; this is impossible since u has a local maximum at q so all of its first derivatives are zero there. This proves that Ω_M is open.

For part *c*), write $c^-(x) = \min(c(x), 0)$. Then u satisfies $Lu + c^+u = -c^-u \le 0$. If u were zero somewhere, it would have a local maximum there and hence be a constant by part *b*). □

REMARK 2.1 The function $u(x) = \sin x$ satisfies $-u'' - u = 0$ on the interval $\Omega = \{0 < x < \pi\}$ and has a positive maximum. This shows that either the assumption on $c(x)$ — or some related assumption—is needed in part *b)*. This is a special case of the following. Let φ be an eigenfunction and λ the corresponding eigenvalue of the operator L on Ω with the Dirichlet boundary condition $\varphi = 0$ on $\partial\Omega$ (for a compact manifold without boundary there is no boundary condition). Then

$$(L - \lambda)\varphi = 0.$$

Then φ (or $-\varphi$) has an interior positive maximum, also illustrating the need for some condition such as on the sign of c. This also proves that the eigenvalues of L with Dirichlet boundary conditions are positive.

In addition, one can use this example as motivation we can replace the sign condition on c by the optimal condition for part *b*) to hold; this condition is that the lowest eigenvalue, λ_1 of the operator $L + c$ is at most zero, $\lambda_1 \le 0$. If the boundary of Ω is smooth, then one can let $v = u/\varphi_1$ and apply the strong maximum principle to v. This has been clarified and the maximum principle extended to situations where the boundary of Ω is not smooth in [BNP].

Under the assumptions of part *b*) a *negative* maximum may occur. For instance, the function $u(x) = -\cosh x$ satisfies $u'' - u = 0$ and has a negative maximum at $x = 0$.

The next Corollary is a standard application of the maximum principle.

Corollary 2.10 *Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu + cu = 0$ in a bounded domain Ω with $c(x) \ge 0$. Then*

$$\max_{\Omega}|u| \le \max_{\partial\Omega}|u| \tag{2.19}$$

Proof. Since $\overline{\Omega}$ is compact, $|u|$ attains its maximum somewhere. Replacing u by $-u$ if needed, we may assume that $u \ge 0$ there. If u is not a constant, this cannot be at an interior point of Ω. □

REMARK 2.2 If one deletes the assumption that $u \in C^(\overline{\Omega})$, the this Corollary is still true if in equation 2.19 we replace *max* by *sup*.

Corollary 2.11 *If u satisfies $Lu + cu = 0$ with $c(x) \ge 0$ on a compact manifold M without boundary, then u must be a constant. If $c(x) > 0$ somewhere, then $u \equiv 0$.*

The most typical and most important application of the maximum principle is to compare solutions of related problems.

Theorem 2.12 [COMPARISON PRINCIPLE] *Let Ω be a bounded domain and $c \ge 0$ there. If u and v are in $C^2(\Omega) \cap C(\overline{\Omega})$ and satisfy $Lu + cu \le Lv + cv$ in Ω with $u \le v$ on $\partial\Omega$, then $u \le v$ throughout Ω.*

In particular, if $Lu + cu = Lv + cv$ in Ω with $u = v$ on $\partial\Omega$, then $u = v$ throughout Ω.

Proof. Let $w = u - v$. Then $Lw + cw \le 0$ in Ω. Either w is a constant or else it cannot have a local non-negative maximum. In either case, since $w \le 0$ on $\partial\Omega$, we conclude that $w \le 0$. □

EXAMPLE 2.10 [UNIQUENESS OF THE DIRICHLET PROBLEM] One immediate consequence is the uniqueness of the Dirichlet problem

$$Lu + cu = f \quad \text{in} \quad \Omega \qquad \text{with} \quad u = \varphi \text{ on } \quad \partial\Omega, \tag{2.20}$$

assuming Ω is bounded and $c \ge 0$. If Ω is unbounded, the assertion is false; for instance $u(x, y) = y$ is harmonic in the upper half plane $\{y > 0\}$ and zero on the boundary.

EXAMPLE 2.11 [CONTINUOUS DEPENDENCE] Another immediate consequence is that in the uniform norm, the solution u of the Dirichlet problem 2.20 in a bounded domain Ω with $c \ge 0$ depends continuously on the boundary data φ. Indeed, if u is a solution of equation (2.20) while v is a solution of the same equation but with $v = \psi$ on $\partial\Omega$ where $|\varphi - \psi| < \epsilon$, then applying the estimate (2.20) to $w = u - v$ we conclude that $|u - v| < \epsilon$ throughout Ω. □

The example in Remark 2.1 of eigenfunctions of L which are zero on $\partial\Omega$ also shows that some condition, such as the sign assumption on c is needed for uniqueness.

REMARK 2.3 Stampacchia proved a version of the maximum principle that is appropriate for certain elliptic equations whose coefficients need not be continuous and for "weak" solutions. Here is the essential idea in the special case of the Laplacian. Say u is a solution of $-\Delta u + cu \ge 0$ in a bounded set Ω with $u = 0$ on the boundary. Let $v \in H^{2,1}$ be non-negative. Then, multiplying the equation by v and integrating by parts, we see that

$$\int_\Omega (\nabla u \cdot \nabla v + cuv)\, dx \ge 0 \tag{2.21}$$

For the next step we use the fact that if $\Psi(s)$ is Lipshitz continuous, and $z(x)$ is in $H^{2,1}$, then $\Psi(z(x))$ is also in $H^{2,1}$ (see [GT]). In partcular $v(x) = \max(u(x), 0) = \frac{1}{2}(u + |u|) \in H^{2,1}$. With this choice of v the previous formula becomes

$$\int_{u \ge 0} (|\nabla v|^2 + cv^2)\, dx \ge 0.$$

If $c \ge 0$ this implies that $\nabla v = 0$. Since $u = 0$ on the boundary of Ω, we conclude that $v \equiv 0$ and hence that $u \le 0$ throughout Ω.

This proof uses only that $u \in H^{2,1}$ satisfies equation (2.21) for all non-negative $v \in H^{2,1}$. In the language of Section 4.2 below, u is a weak solution of $-\Delta u + cu \ge 0$. □

2.7 Proving the Index Theorem

There are several different proofs of the index theorem. We will give the first step of one approach using the heat equations

$$u_t = -L^* L u \qquad \text{and} \qquad u_t = -L L^* u,$$

where L is a linear elliptic operator. Let K_1 and K_2 be the heat kernels associated with these two equations (see (1.3), (1.12)–(1.14)). We claim that

$$i(L) = \int [K_1(y,y;t) - K_2(y,y;t)]\, dy \tag{2.22}$$

for all $t > 0$, that is,

$$i(L) = \operatorname{trace}(e^{-tL^*L} - e^{-tLL^*}) = \sum_j (e^{\lambda_j t} - e^{-\mu_j t}), \tag{2.23}$$

where λ_j are the eigenvalues of L^*L and μ_j those of LL^*. Notice that if $\lambda_j \neq 0$ is an eigenvalue of L^*L, then it is also an eigenvalue of LL^* (since $L^*L\varphi = \lambda\varphi$ then $LL^*(L\varphi) = \lambda(L\varphi)$). Also, the multiplicity of the eigenvalue $\lambda = 0$ of L^*L is $\dim\ker L^*L = \dim\ker L$, with a similar statement for $\mu = 0$. Therefore the non-zero eigenvalue terms in (2.23) all cancel, while the zero eigenvalue terms give the index of L.

Having the formula (2.22) for the index one needs other properties of the heat kernels K_1 and K_2 to obtain a formula expressing the integrand in (2.22) in terms of topological information such as characteristic classes of the manifold. There are several ways of doing this. One method investigates the asymptotic behavior of the heat kernels as $t \to 0$ or $t \to \infty$. See [At] for further discussion.

2.8 Linear Parabolic Equations

Since we will be using parabolic equations to solve some elliptic equations, we must collect a few of the basic facts. The simplest parabolic equation is the *heat equation*

$$\frac{\partial u}{\partial t} = \Delta u \tag{2.24}$$

on a compact manifold M. We may think of $u(x,t)$ as the temperature at the point $x \in M$ at time t. The *initial value problem* (or *Cauchy problem*) is: given the initial temperature distribution,

$$u(x,0) = \varphi(x), \tag{2.25}$$

find the temperature, $u(x,t)$ for all $t > 0$. Thus, solve (2.24)-(2.25).

More generally we can replace (2.24) by

$$\frac{\partial u}{\partial t} = Lu + f(x,t) \tag{2.26}$$

where L is a linear strongly elliptic differential operator (see [F, Part 2.9], [H-1], and [LSU, Chapter IV.5 and Chapter VII]).

Many of the results for elliptic equations have related versions for parabolic equations of the type (2.26). But first we must define what a parabolic equation is. To avoid a long discussion, we will limit ourselves to a special case, which, however, will be adequate us. In local coordinates consider *linear* systems of the form

$$\frac{\partial u}{\partial t} - Lu := \frac{\partial u}{\partial t} - \sum_{i,j} A^{ij}(x,t)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum B_i(x,t)\frac{\partial u}{\partial x^i} + C(x,t)u, \tag{2.27}$$

where $u = (u^1, \dots, u^N)$ is a vector, the A^{ij}, B_i and C are $N \times N$ matrices, and for each ij, the matrix A^{ij} has the form $A^{ij}(x,t) = a^{ij}(x,t)I$. The equation (2.27) is *parabolic* at (x,t) if the matrix $(a^{ij}(x,t))$ is positive definite. The equation $u_t = \Delta u$ on a tensor field u is of this form.

Before presenting anything difficult, we state a version of the *maximum principle* for scalar parabolic operators of the form

$$Pu = u_t - \sum \left(a^{ij}u_{ij} + b^i u_i + cu\right), \tag{2.28}$$

where $u_i = \partial u/\partial x_i$ etc., and the coefficients $a^{ij}(x,t)$, b_i, and c are continuous with $(a^{ij}(x,t))$ positive definite.

Theorem 2.13 [Maximum Principle] *Assume $u(x,t)$ is a smooth function satisfying $Pu \le 0$ for $x \in M$ and $0 \le t \le T$. If $c \le 0$, then u can not have a positive maximum at a point (x,t) with $t > 0$ unless $u \equiv$ constant. In particular, $u(x,t) \le \max|u(x,0)|$.*

See [PW] for a proof. There is an obvious proof if $c < 0$. First make a linear change of variables so that at this point u_{ij} is diagonal. Then use the fact that at an interior positive maximum $u_t = u_i = 0$ while the hessian, u_{ij} is positive semidefinite. The case where $c(x) \le 0$ needs an extra trick we forgo.

A useful variant of this is in [H-1, p. 101]. As an easy corollary we have a uniqueness result for the initial value problem

$$Pu = f(x,t) \quad \text{for} \quad t > 0 \qquad \text{with} \quad u(x,0) = \varphi(x). \tag{2.29}$$

Corollary 2.14 [Uniqueness] *There is at most one solution of* (2.29).

Proof. To prove this, say $Pu = 0$ with $u(x,0) = 0$ for $0 < t \le T$. First assume $c < 0$ in (2.28). Then by the maximum principle above, $u(x,t) \le 0$. Applying the maximum principle to $-u$ we find that $u(x,t) \ge 0$. Thus $u(x,t) \equiv 0$ for all $0 \le t \le T$. To reduce to the case $c < 0$ we use a standard device and let $u(x,t) = v(x,t)e^{\alpha t}$, where α is a constant to be determined. Then

$$0 = Pu = e^{\alpha t}[Pv + \alpha v] \quad \text{with} \quad v(x,0) = 0. \tag{2.30}$$

By choosing α appropriately, v satisfies an equation of the form (2.28) with $c < 0$. The previous reasoning applies to show $v = 0$ and hence $u = 0$. □

In order to state the basic existence theorem and related inequalities, we introduce the appropriate spaces for (2.27). These are needed because (2.27) is second order in the space variables x, but only first order in time t. For an open set $\Omega \subset \mathbb{R}^n$, let $Q_T = \Omega \times [0,T]$ for some $T > 0$, let $\delta = k + \alpha$, where $k \ge 0$ is an integer and $0 < \alpha < 1$, and define $C^{\delta,\delta/2}(Q_T)$ to be the Banach space of functions $u(x,t)$ whose derivatives $\partial_t^r \partial_x^s u$ (here s is a multi-index) are continuous for $2r + |s| \le k$ and have finite norm

$$\|u\|_{\alpha,\alpha/2} = \sum_{2r+|s|\le k} \|\partial_t^r \partial_x^s u\|_{C(Q_T)} + \max_{2r+|s|=k} [\partial_t^r \partial_x^s u]_{\alpha,\alpha/2}, \tag{2.31}$$

where, as in (1.17)

$$[v]_{\alpha,\beta} = \sup_{0\le t\le T}\Big([v,(\cdot,t)]_{\alpha,\Omega}\Big) + \sup_{x\in\Omega}\Big([v(x,\cdot)]_{\beta,[0,T]}\Big)$$

is a Hölder norm with exponent α in the space variable and β in the time variable. One defines these spaces on a manifold—or vector bundle—using a partition of unity in the usual manner.

For a linear parabolic operation L of the form (2.27), we next state the main result for the initial value problem

$$u_t - Lu = f(x,t) \quad \text{for} \quad t > 0 \qquad \text{with} \quad u(x,0) = \varphi(x) \tag{2.32}$$

assuming the coefficients of L belong to $C^{\alpha,\alpha/2}(Q_T)$ (with $\delta = k + \alpha$ as above, but now $Q_T = M \times [0, T]$).

Theorem 2.15 *For any* $f \in C^{\delta,\delta/2}(Q_T)$ *and any* $\varphi \in C^{\delta+2}(M)$ *, there is a unique solution* $u \in C^{\delta+2,\delta/2+1}(Q_T)$ *of* (2.32)*. Moreover,* u *satisfies the basic inequality*

$$\|u\|_{\delta+2,\delta/2+1} \le c_1(\|u_t - Lu\|_{\delta,\delta/2} + \|u(\cdot,0)\|_{\delta+2}), \tag{2.33}$$

where the constant c_1 *is independent of* u *(but does depend on coefficients of* L *), and the first two norms are over* Q_T *, while the last norm is over* M *.*

A proof of this for a single equation is in [LSU, Theorem 5.2, p. 320]; a related proof for systems is in Chapter VII of the same reference. One can prove a similar result using Sobolev spaces in place of the Hölder spaces ([LSU, Theorem 9.1, p. 341-342] and [H-1 p. 120-121]).

Chapter 3

Geometric Applications of Linear Elliptic Operators

3.1 Introduction

We will give a few standard examples where linear elliptic equations arise in geometry. For a first reading, we suggest that one work with the two-dimensional flat torus. The smooth functions on this torus are then the smooth doubly periodic functions in ordinary Euclidean space so no geometric complications arise. It is amazing that one can build such a rich theory of Riemannian manifolds using only the slender assumption that they are locally like Euclidean space, except that one permits a more flexible way to measure arc length.

3.2 Hodge Theory

a) Hodge Decomposition

One obtains the classical Hodge decomposition theorem for a real compact connected orientable Riemannian manifold M^n without boundary as an immediate consequence of the Fredholm alternative, in particular, part (a) of Corollary 2.5, applied to the *Hodge Laplacian*

$$\Delta_H = dd^* + d^*d \tag{3.1}$$

(see (2.4)) acting on the space $\Omega^p(M)$ of smooth, i.e. C^∞, differential p-forms. Note that $d^{*2} = 0$ simply because $d^2 = 0$. Introduce the space $\mathcal{H}^p$ of *harmonic p-forms*, where we define $\mathcal{H}^p = \ker \Delta_H$ acting on $\Omega^p(M)$. This space $\mathcal{H}^p$ is finite dimensional by Theorem 2.4. Also, $h \in \mathcal{H}^p$ if and only if h is both closed ($dh = 0$) and co-closed ($d^*h = 0$) because

$$\langle h, \Delta_H h\rangle = \langle h, dd^*h + d^*dh\rangle = \|d^*h\|^2 + \|dh\|^2$$

If $\varphi \in \Omega^p(M)$, then by picking an orthogonal basis for $\mathcal{H}^p$ we can decompose φ as the orthogonal sum $\varphi = \psi + h$, where $h \in \mathcal{H}^p$ and $\psi \perp \mathcal{H}^p$. Part (a) of Corollary 2.5 shows there is a solution ω of $\Delta_H\omega = \psi$. Thus

$$\varphi = \Delta_H\omega + h = dd^*\omega + d^*d\omega + h,$$

that is,

$$\Omega^p(M) = \text{image}\left\{\Delta_H(\Omega^p(M))\right\} \oplus \mathcal{H}^p,$$

with the terms on the right being orthogonal. We can rewrite the above decomposition of φ as

$$\varphi = d\alpha + d^*\beta + h, \tag{3.2}$$

where $\alpha = d^*\omega$ and $\beta = d\omega$. Observe $d^2 = 0$ implies that $d\alpha \perp d^*\beta$ because

$$\langle d\alpha,\, d^*\beta\rangle = \langle d^2\alpha,\, \beta\rangle = 0$$

Similarly, $d\alpha \perp h$ and $d^*\beta \perp h$. Equation (3.2) is the *Hodge decomposition* of an arbitrary p-form into the orthogonal sum of exact, co-exact, and harmonic forms.

As a special case, we use the Hodge decomposition when φ is closed. Then applying d to (3.2) gives $0 = d\varphi = dd^*\beta$, which implies $\|d^*\beta\|^2 = \langle \beta,\, dd^*\beta\rangle = 0$, so $d^*\beta = 0$. Hence we can write a closed form as

$$\varphi = d\alpha + h.$$

Thus, h is the unique harmonic form in the same de Rham cohomology class as the closed form φ. This proves that the space $\mathcal{H}^p$ of harmonic p-forms is isomorphic to the de Rham p-cohomology group; it also implies that the de Rham group H^p_{deR} is finite-dimensional, a fact that otherwise hardly is obvious. If β_p is the p^{th} Betti number, then

$$\beta_p = \dim H^p_{deR} = \dim \mathcal{H}^p \tag{3.3}$$

b) Poincaré Duality

We next prove Poincaré duality. This proof uses the *Hodge star operator*, $\star$, which maps a p–form to an $n-p$-form; it is defined at every point of an oriented Riemannian manifold M^n by using the pointwise inner product of p-forms α and β:

$$\alpha \wedge \star\beta = (\alpha, \beta)\, dx_g.$$

A special case is $\star 1 = dx_g$. One uses the orientation of M to define dx_g globally. The global and local inner products of p-forms are related by $\langle \varphi,\, \psi\rangle = \int_{\mathcal{U}} (\varphi, \psi)\, dx_g$. From these one can verify the following properties.

a). SQUARE: $\star\star = (-1)^{p(n-p)}$.
The proof of this depends critically on the detailed construction of the pointwise inner product on p-forms. One approach is, in an n dimensional inner product space V, extend this inner product to the exterior algebra $\Lambda(V) = \oplus\Lambda_p(V)$ by saying that if $e_1, \ldots, e_n$ are orthonormal vectors then the p-vectors $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}\}$ with $i_1 < i_2 < \ldots < i_p$ are an orthonormal basis for $\Lambda_p(V)$.

b). ISOMETRY: In the pointwise inner product, $(\star\alpha, \star\beta) = (\alpha, \beta)$.
If $\alpha \in \Lambda_p(V)$ and $\gamma \in \Lambda_{n-p}(V)$, since $\alpha \wedge \gamma = (-1)^{p(n-p)}\gamma \wedge \alpha$, then

$$\begin{aligned}(\alpha,\, \star\gamma)dx_g =& (\star\gamma,\, \alpha)dx_g = \star\gamma \wedge \star\alpha = (-1)^{p(n-p)} \star \alpha \wedge \star\gamma \\ =& (-1)^{p(n-p)}(\star\alpha,\, \gamma)dx_g.\end{aligned}$$

Applying this with $\gamma = \star\beta$ for $\beta \in \Lambda_p(V)$ and using $\star\star = (-1)^{p(n-p)}$ we obtain the desired isometry

$$(\alpha,\, \beta)dx_g = (-1)^{p(n-p)}(\alpha,\, \star\star\beta)dx_g = (\star\alpha,\, \star\beta)dx_g.$$

c). ADJOINT OF d: $\quad d^* = (-1)^{n(p+1)+1} \star d\star$.

The computation of the adjoint goes as follows. For any open set $\mathcal{U}$ and any $\alpha \in \Omega^{p-1}(\mathcal{U})$ and $\beta \in \Omega^p(\mathcal{U})$, then pointwise

$$\begin{aligned} d(\alpha \wedge \star\beta) &= d\alpha \wedge \star\beta + (-1)^{p-1}\alpha \wedge d \star \beta \\ &= d\alpha \wedge \star\beta - (-1)^{n(p+1)+1}\alpha \wedge \star\star\, d \star \beta \\ &= (d\alpha, \beta)\, dx_g - (\alpha, (-1)^{n(p+1)+1} \star d \star \beta)\, dx_g. \end{aligned}$$

We integrate both sides of the above formula over $\mathcal{U}$ and assume that α and β vanish outside of $\mathcal{U}$. Using Stokes' theorem we are done since the formal adjoint, d^*, is defined by the property $\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle$.

COMMUTES WITH LAPLACIAN: $\Delta_H \star = \star \Delta_H$,
This follows from $\Delta_H = d^*d + dd^*$ and the above formula for d^*.

These imply that $\star$ is an isometry and maps harmonic forms to harmonic forms. Thus $\star : \mathcal{H}^p \to \mathcal{H}^{n-p}$ is an isometry. In particular $\dim \mathcal{H}^p = \dim \mathcal{H}^{n-p}$, which is called *Poincaré duality*.

c) The de Rham Complex

As a final exercise using the Hodge Laplacian, we introduce the direct sum $\Omega(M)$ of the space of all smooth differential forms, $\Omega(M) = \oplus_{0\le p\le n}\Omega^p(M)$ and give it the obvious inner product by simply requiring that the various Ω^p be orthogonal. Then we define the differential operator $d : \Omega(M) \to \Omega(M)$ by having it act on each term in the usual way; in particular, $d^2 = 0$. The adjoint, d^* is computed on each term Ω^p just as above and also satisfies $d^{*2} = 0$.

One sees immediately that $(d+d^*)^2 = dd^* + d^*d = \Delta_H$ is the Hodge Laplacian. This implies that

$$D := d + d^* : \Omega(M) \to \Omega(M) \tag{3.4}$$

is a first order elliptic differential operator, which one can think of as the square root of Δ_H acting of $\Omega(M)$. One often refers to $\Omega(M)$ with the operator D as the *de Rham complex.*

It is easy to see that $\ker(d + d^*) = \ker \Delta_H$ is the space of harmonic forms; one just imitates the above proof that harmonic forms are both closed and co-closed.

Because D is self-adjoint, its index (see (2.2)) is zero. To obtain an operator with an interesting index, we consider the odd and even parts of $\Omega(M)$ separately. Let

$$\Omega^{\text{even}}(M) = \bigoplus_{p \text{ even}} \Omega^p(M) \qquad \text{and} \qquad \Omega^{\text{odd}}(M) = \bigoplus_{p \text{ odd}} \Omega^p(M)$$

and let $D^+ : \Omega^{\text{even}} \to \Omega^{\text{odd}}$ be the restriction of D to Ω^{even}. Similarly we define $D^- : \Omega^{\text{odd}} \to \Omega^{\text{even}}$ and note $D^- = (D^+)^*$. D^+ is elliptic (because D is). Also $\ker D = \ker D^2$ (if $D^2u = 0$ then $0 = \langle u, D^2u\rangle = \|Du\|^2$ so $Du = 0$). If we let $\mathcal{H}^p$ be the space of harmonic p-forms, then

$$\ker D^+ = \bigoplus_{p \text{ even}} \mathcal{H}^p, \qquad \ker D^- = \bigoplus_{p \text{ odd}} \mathcal{H}^p.$$

Because $\dim \mathcal{H}^p = \beta_p$, we obtain

$$\text{index } D^+ = \sum (-1)^p \dim \mathcal{H}^p = \chi(M), \tag{3.5}$$

where $\chi(M)$ is the Euler characteristic. By Poincaré duality, which we proved just above, we note that for odd dimensional manifolds M this is zero .

More generally, if one asks for the square root of the Laplacian on other vector spaces, one is led to the various Dirac operators; another special case of the Dirac operator is the Cauchy-Riemann operator. We discuss this a bit more in Section 3.5 below.

3.3 Eigenvalues of the Laplacian

On a smooth compact Riemannian manifold (M, g) without boundary, the Laplacian is formally self-adjoint. It has eigenfunctions and eigenvalues with all of the same formal properties as the eigenvectors and eigenvalues of a symmetric matrix. We will apply the machinery of the previous chapter to carry out the proofs. We will treat the Laplacian on functions; at the end we will remark how to extend this to the case of the Laplacian on p-forms. Note that in the case of functions, the Laplacian has the *opposite* sign from the convention we have used for the Hodge Laplacian on p-forms. Thus, in this section on $\mathbb{R}^1$ here we have $\Delta u = +u''$. Also, since we will only be working in an L^2 setting, we will write the Sobolev spaces $H^{2,k}$ simply as H^k.

Theorem 3.1 *On a smooth compact manifold without boundary, the Laplacian acting has an infinite sequence of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \dots$ and corresponding smooth orthonormal eigenfunctions φ_j*

$$-\Delta\varphi_j = \lambda_j\varphi_j. \tag{3.6}$$

Moreover, the eigenspaces are finite dimensional, the eigenvalues have no finite accumulation point and the eigenfunctions are complete in L^2. If f is smooth, then its eigenfunction expansion converges uniformly in C^k for all k.

Step 1 $\lambda_j \ge 0$.

This follows by multiplying (3.6) by φ_j and integrating by parts:

$$\int_M |\nabla\varphi_j|^2\,dx_g = -\int_M \varphi_j\,\Delta\varphi_j\,dx_g = \lambda_j\int_M |\varphi_j|^2\,dx_g = \lambda_j. \tag{3.7}$$

If $\varphi = 1$ then the left side is zero and we see that the lowest eigenvalue is $\lambda_0 = 0$ and the eigenfunction must be a constant. Since this eigenvalue is so obvious, it is usually called the "trivial eigenvalue" but *caution*: since it is the first eigenvalue, is sometimes labeled λ_1.

Step 2 *The eigenspaces are finite dimensional and the eigenvalues have no finite accumulation point. Thus $\lambda_k \to \infty$.*

Let $\mathcal{S}_m = \{\varphi_j\}$ be an orthonormal set of all the eigenfunctions with eigenvalues $\lambda_j \le m$. We will show that $\mathcal{S}$ is finite dimensional. From the formula (3.7) in Step 1 we see that for some constant c

$$\int_M |\nabla\varphi_j|^2\,dx_g = \lambda_j \le m.$$

Thus the φ_j lie in a bounded set in H^1. By the Sobolev embedding theorem 1.1, they thus are in a relatively compact set in L^2. But if this set had infinitely many elements one could find an L^2 convergent subsequence which would contradict the orthogonality.

Step 3 *Eigenfunctions corresponding to different eigenvalues are orthogonal.*

The proof for matrices works here too. One needs the fact that the Laplacian is formally self-adjoint.

STEP 4 *Existence of Eigenfunctions.*

Proceeding inductively, say we already have the eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_k$, each eigenvalue repeated to the dimension of its eigenspace. Let $\mathcal{E}_k = \{\varphi_j,\ j = 0, \ldots, k\}$ be the corresponding set of smooth eigenfunctions. We will prove the existence of λ_{k+1} and corresponding smooth eigenfunction. The proof uses techniques from the calculus of variations, which we will again apply in Section 5.2. It also closely follows the standard linear algebra technique for obtaining the successive eigenvalues of a positive quadratic form. We will write $\mathcal{E}_k^\perp$ for the H^1 functions in the L^2 orthogonal complement of $\mathcal{E}_k$ and all unspecified norms will be L^2 (since smooth functions are dense, this is equivalent to taking the H^1 closure of the smooth functions orthogonal to $\mathcal{E}$). Observe that, by self-adjointness, Δ maps $\mathcal{E}_k^\perp$ to itself. Based on the linear algebra case, and using (3.7) we are led to believe that the next eigenvalue, λ_{k+1}, will be

$$\lambda_{k+1} = \lambda := \inf_{u \in \mathcal{E}_k^\perp} \frac{\langle u,\, -\Delta u\rangle}{\|u\|^2} = \inf_{u \in \mathcal{E}_k^\perp} \frac{\|\nabla u\|^2}{\|u\|^2}. \tag{3.8}$$

The fraction on the right is called the Rayleigh (or Rayleigh-Ritz), quotient.

Since $\mathcal{E}_{k-1} \subset \mathcal{E}_k$, we immediately know that $\lambda \ge \lambda_k$. We will show that λ is the desired eigenvalue λ_{k+1}.

Let $u_j \in \mathcal{E}_k^\perp$ be a sequence with $\|u_j\| = 1$ so that $\|\nabla u_j\|^2 \to \lambda$. Then there is a constant c_1 so that $\|\nabla u_j\| < c_1$. Because the H^1 norm satisfies $\|f\|_{H^1}^2 = \|\nabla f\|^2 + \|f\|^2$, the $\{u_j\}$ are in a bounded set in H^1.

Bounded sets in (infinite dimensional) Hilbert spaces $\mathcal{H}$, such as H^1, are not usually compact, but they are weakly compact. To define this, we say that a sequence $x_j \in \mathcal{H}$ *converges weakly* to some $x \in \mathcal{H}$ if the numerical sequence $\langle x_j,\, z\rangle \longrightarrow \langle x,\, z\rangle$ converges for any z in $\mathcal{H}$. We write $x_j \rightharpoonup x$ for weak convergence. The standard example is that an orthonormal basis converges weakly to zero. *Weak compactness* means that any bounded sequence $\{x_j\}$ has a subsequence (which we relabel x_j) that converges to some element x of the Hilbert space.

The above example of an orthonormal basis also shows that the norm is not continuous under weak convergence, but it is *lower semicontinuous*, that is, $\|x\| \le \liminf \|x_j\|$, which is enough for many applications—including ours. The short proof that the norm is lower semicontinuous under weak convergence is as follows. Say $x_j \rightharpoonup x$. Then

$$\|x\|^2 = \lim \langle x,\, x_j\rangle \le \liminf \|x\|\, \|x_j\|. \tag{3.9}$$

Returning to our eigenvalue problem, since the $\{u_j\}$ are in a bounded set in H^1, there is a subsequence, which we relabel u_j, that converges weakly to some v in H^1. Moreover, by the Sobolev embedding theorem (actually, an older result due to Rellich) the embedding of H^1 in L^2 is compact, and, as is easily proved, in a Hilbert space, compact operators map weakly convergent sequences to sequences that converge in norm. Thus the sequence u_j also converges to u in L^2. This implies that $\|u\| = 1$ and, by lower semicontinuity,

$$\|\nabla u\| \le \liminf \|\nabla u_j\| = \lambda$$

However, from (3.8) we also know that $\|\nabla u\| \ge \lambda$. Therefore $\|\nabla u\| = \lambda$ so u gives the desired minimum. Also, the weak convergence in H^1 implies that for any v in H^1 we have

$$0 = \lim\, [\langle \nabla u_j,\, \nabla v\rangle - \lambda \langle u_j,\, v\rangle] = [\langle \nabla u,\, \nabla v\rangle - \lambda \langle u,\, v\rangle], \tag{3.10}$$

that is,

$$0 = \int_M [\nabla u \cdot \nabla v - \lambda u v]\, dx_g. \tag{3.11}$$

for any v in H^1. If we knew that u were twice differentiable, we could integrate this by parts and conclude that

$$\int_M (\Delta u + \lambda u)\, v\, dx_g = 0$$

for all v in H^1, and hence that $\Delta u + \lambda u = 0$ so u would be the desired eigenfunction. Because of this, a function u that satisfies (3.11) for all v in H^1 is called a *weak solution* of $\Delta u + \lambda u = 0$.

To show that this weak solution is a smooth solution, we us the observation that the null space of $Lw := -\Delta w + w$ is zero (see Example 2.9). Thus by the Fredholm alternative, since $u \in H^1$ there is a unique solution $w \in H^3$ of $Lw = (1+\lambda)u$. We claim that $u = w$, and hence that $u \in H^3$. To prove that $u = w$, let $z = w - u \in H^1$. Since u is a weak solution of $Lw = (1+\lambda)u$, then z is a weak solution of $Lz = 0$, that is

$$\int_M [\nabla z \cdot \nabla v + zv]\, dx_g = 0$$

for all $v \in H^1$. In particular, letting $v = z$, we see that $z = 0$. This proves that the weak solution u is in H^3. Now we can use the bootstrap procedure of Example 2.5 to conclude that u is the desired smooth eigenfunction. We label this eigenfunction φ_{k+1}.

Step 5 L^2 *Completeness of the Eigenfunctions.*

Let P_N be the (self-adjoint) orthogonal projection onto the eigenspace spanned by the first N eigenfunctions

$$P_N f = \sum_{j \le N} \langle f, \varphi_j \rangle \varphi_j.$$

We wish to show that the $f - P_N f \to 0$ as $N \to \infty$. We will first prove this for any $f \in H^2$. By definition of λ_{N+1} (see (3.8)) we know that for any $f \in H^2$

$$\|f - P_N f\|^2 \le \frac{1}{\lambda_{N+1}} \langle \Delta(f - P_N f),\, f - P_N f \rangle. \tag{3.12}$$

Now it is easy to verify, just as in linear algebra, that on functions in H^2 we have $\Delta P_N = P_N \Delta$. Thus, if $f \in H^2$ we know $\langle \Delta P_N f,\, f - P_N f \rangle = 0$ so that

$$\begin{aligned} \langle \Delta(f - P_N f),\, f - P_N f \rangle &= \langle \Delta f,\, f - P_N f \rangle \\ &= \|\nabla f\|^2 - \|\nabla P_N f\|^2 \le \|\nabla f\|^2. \end{aligned}$$

In equation (3.12), use this and the fact that $\lambda_{N+1} \to \infty$ to prove completeness for functions in H^2. This also shows that

$$\|f\|^2 - \|P_N f\|^2 = \|f - P_N f\|^2 \longrightarrow 0.$$

To extend this completeness proof to all functions $f \in L^2$, we use that smooth functions are dense in L^2. Thus, there is a smooth function h so that $\|f - h\| < \frac{1}{2}\epsilon$. Pick N so that $\|h - P_N h\| < \frac{1}{2}\epsilon$. Since $(f - P_N f) \perp P_N(f - h)$ for any h, the Pythagorean theorem gives the "best L^2 approximation property"

$$\|f - P_N f\|^2 = \|f - P_N h\|^2 - \|P_N(f - h)\|^2 \le \|f - P_N h\|^2. \tag{3.13}$$

Therefore

$$\|f - P_N f\| \le \|f - P_N h\| \le \|f - h\| + \|h - P_N h\| < \epsilon.$$

Step 6 L^2 *Uniform Convergence of the Eigenfunction Expansion*

For smooth functions f, their eigenfunction expansions converge in C^j for all j, while for less smooth functions, the convergence is only for certain j. By the Sobolev Inequality (1.26), this will be a consequence of the more precise statement that *if* $f \in H^k$ *then the eigenfunction expansion converges in* H^k. In Step 5 above we did the case $k = 0$.

The essential ingredient is that for smooth functions we can use the invertible elliptic operator $-\Delta + I$ (which we also used above) to define a norm equivalent to the H^k norm. Informally, we would like to define a first order operator $Q = (-\Delta + I)^{1/2}$ and then let

$$\langle u,\, v\rangle_{H^k} = \langle Q^k u,\, Q^k v\rangle.$$

While one can develop a useful formalism to define such an operator Q, an *ad hoc* procedure is adequate for our immediate needs. For smooth functions the above formula gives

$$\langle u,\, v\rangle_{H^k} = \langle u,\, (-\Delta + I)^k v\rangle. \tag{3.14}$$

and we can treat the cases k even and k odd separately.

k even. If k, $k = 2\ell$, we define the inner product to be

$$\langle u,\, v\rangle_{H^k} = \langle (-\Delta + I)^\ell u,\, (-\Delta + I)^\ell v\rangle.$$

so the corresponding norm is

$$\|u\|_{H^k} = \|(-\Delta + I)^\ell u\|.$$

The basic L^2 inequalities for elliptic operators in Theorem 2.2 tells us that this definition of the norm is equivalent to any other.

k odd If $k = 2\ell + 1$ one can use the k *even* case to define the inner product

$$\langle u,\, v\rangle_{H^k} = \langle (-\Delta + I)^\ell u,\, (-\Delta + I)^\ell v\rangle + \langle \nabla(-\Delta + I)^\ell u,\, \nabla(-\Delta + I)^\ell v\rangle.$$

so

$$\|u\|^2_{H^k} = \|(-\Delta + I)^\ell u\|^2 + \|\nabla(-\Delta + I)^\ell u\|^2 = \|u\|^2_{H^{2\ell}} + \|\nabla u\|^2_{H^{2\ell}}$$

For smooth functions, if k is even or odd these agree with (3.14). Thus, for a smooth function f, let $u = (-\Delta + I)^k f$. Since $\Delta P_N = P_N \Delta$, the Schwarz inequality applied to (3.14) then gives

$$\|f - P_N f\|^2_{H^k} \le \|f - P_N f\|\, \|u - P_N u\|.$$

and we know the last term tends to zero by the L^2 completeness (in fact, both factors tend to zero). If f is not smooth but only in some H^k, then we can approximate it by a smooth function h just as at the end of Step 5, only replacing (3.13) by the same assertion in the H^k norm; here we again use $(f - P_N f) \perp P_N(f - h)$, only this time in the H^k inner product. This completes the proof. □

For p-forms with $p \ge 1$, as we saw in the previous section, the eigenspace corresponding to zero eigenvalue is just the space $\mathcal{H}^p$ of harmonic p-forms. Its dimension is a topological invariant. There are several ways to prove the above theorem concerning the eigenvalues and eigenfunctions. One is to use the *Weitzenböck formulas* (see (**??**), which states that the Laplacian on the space Ω^p of smooth p-forms can be written as[1]

$$\Delta\alpha = \nabla^*\nabla\alpha - \mathcal{R}\alpha,$$

[1]Recall that for this Hodge Laplacian we reverse the sign, so in the case of functions on $\mathbb{R}$ it is $-u''$.

where ∇^* is the formal adjoint of ∇ and $\mathcal{R} : \Omega^p \to \Omega^p$ is an expression only involving the curvature of the manifold. In the next section we shall use this in the special case of 1-forms, where $\mathcal{R}$ is simply the Ricci curvature. Using this formula the above proof goes through without change.

An alternate approach, which one could also use for the case we treated, is to use more systematically the fact that the operator $L = -\Delta + I : H^{k+2} \to H^k$ is invertible. Let G be the inverse operator (we use G since the inverse of Laplace-type operators are frequently named Green's operators). Then λ is an eigenvector of $-\Delta$ if and only if $1+\lambda$ is an eigenvalue of L, which is true if and only if $1/(1+\lambda)$ is an eigenvalue of G. Moreover, G has the same eigenfunctions as does Δ. The usefulness of G is because we can also write

$$G: H^k \xrightarrow{L^{-1}} H^{k+2} \xhookrightarrow{inc} H^k,$$

where inc is the natural inclusion of H^{k+2} in H^k. Since by the Sobolev theorem, this natural inclusion is a compact operator, the self-adjoint operator $G : H^k \to H^k$ is a compact operator. One can then immediately apply the spectral theory of self-adjoint compact operators to G and consequently obtain the spectral information for the Laplacian. The resulting proof is not very different from that given above. We have preferred the more direct approach above since it uses ideas from the calculus of variations, which we will meet again in Chapter 5.2.

3.4 Bochner Vanishing Theorems

Bochner[Bo] made several geometric applications of the uniqueness proofs we gave in Example 2.9. We will give two of them, since both the technique and results are interesting.

a) One-parameter Isometry Groups

The first concerns the existence of a one-parameter family of isometries of a compact Riemannian manifold (M, g). These are maps that do not change the length of any curves. The round sphere, the torus of revolution, in fact, all surfaces of revolution in $\mathbb{R}^3$, have an obvious one-parameter group of isometries. The flat torus also has one-parameter groups of isometries. In all of these cases there are points where the curvature is positive or zero. This is not a coincidence. We will now show that if a compact manifold has negative Ricci curvature, then it cannot have a one parameter group of isometries.

One surprise is that this theorem is not difficult to prove. Say $\varphi_t : (M, g) \to (M, g)$ is a one-parameter family of isometries for $t \in (-\epsilon, \epsilon)$. We will need the *infinitesimal generator* of a one-parameter family of maps $\varphi_t : M \to M$ with $\varphi_t|_{t=0} = \text{id}$, the identity map. The infintessimal generator is the vector field $V = d\varphi_t/dt|_{t=0}$. We begin by observing that if V is the infinitessimal generator of a one-parameter family of isometries, then the Lie derivative, $L_V g := \frac{d}{dt}\varphi_t^*(g) = 0$. In tensor notation we will show that this means

$$V_{i;j} + V_{j;i} = 0, \tag{3.15}$$

where the semicolon ; indicates covariant differentiation. A vector field having the property (3.15) is called a *Killing vector field*. For simplicity, we will prove (3.15) directly without introducing the language of Lie derivatives.

Note that in local coordinates, under the map $x^i = \varphi_t^i(y)$ the metric $g = \sum g_{ij} dx^i dx^j$ becomes

$$\varphi_t^*(g) = \sum g_{ij}(\varphi_t(y)) \frac{\partial \varphi_t^i(y)}{\partial y^k} \frac{\partial \varphi_t^j(y)}{\partial y^\ell} \, dy^k dy^\ell. \tag{3.16}$$

That these maps are isometries implies the derivative of the right-hand side evaluated at $t = 0$ must be zero,

$$\sum_p \left(\frac{\partial g_{kl}}{\partial x^p} \frac{\partial \varphi^p}{\partial t} + \sum_{i,j} g_{ij} \left(V^i{}_{,k} \delta^j_\ell + \delta^i_k V^j{}_{,\ell} \right) \right) = 0$$

for all k, ℓ. In this formula , means partial derivative with respect to the indicated coordinate. To interpret this we use Riemannian normal coordinates at x, so that at the one point x we have $g_{ij} = \delta_{ij}$ and its first derivatives are zero, $\partial g_{kl}(x)/\partial x^p = 0$. In these coordinates partial differentiation and covariant differentiation coincide. Thus the last formula is seen to agree with (3.15).

Theorem 3.2 [Bochner] *If (M, g) is a compact manifold with non-positive Ricci curvature, then any Killing vector field has zero covariant derivative, that is, it is parallel. If in addition the Ricci curvature at one point is negative, then (M, g) has no non-trivial Killing vector fields, so it does not have any one-parameter families of isometries.*

Proof. The insight for discovering this theorem in the first place is, "if you have an interesting object, then taking its Laplacian may give something useful." Say we have a Killing vector field V. Then $|V|^2$ is an interesting scalar-valued function so we compute $\Delta|V|^2$. In tensor notation

$$\Delta|V|^2 = (V^i V_i)_{;j}{}^{;j} = 2V^{i;j} V_{i;j} + 2V^i V_{i;j}{}^{;j}. \tag{3.17}$$

Using the property (3.15) of Killing vector fields we have

$$V_{i;j}{}^{;j} = -V_{j;i}{}^{;j} = -V^j; ij.$$

The Ricci commutation formula (A.45), combined with $V^j{}_{;j} = 0$, which follows from (3.15), gives

$$V^j{}_{;ij} = V^j{}_{;ji} + V^k R_{ki} = V^k R_{ki}.$$

Using these facts in (3.17) we conclude that for a Killing field

$$\Delta|V|^2 = 2|\nabla V|^2 - 2\operatorname{Ric}(V, V), \tag{3.18}$$

where here we view the Ricci curvature as a quadratic form acting on the vector V. At this point we can use either of the two methods used in Example 2.9.

Method 1 Since the Ricci curvature is nonpositive, from (3.18) $\Delta|V|^2 \geq 0$. Thus by the maximum principle 2.9, $|V|$ is a constant and the right side of (3.18) is zero. In particular that $\nabla V = 0$, that is, V is parallel. If the Ricci curvature is negative at one point, then since $\operatorname{Ric}(V, V) = 0$, we must have $V = 0$ at that point and hence everywhere.

Method 2 Integrate this last formula over M:

$$0 = \int_M \Delta|V|^2 \, dx_g = 2\int_M [|\nabla V|^2 - \operatorname{Ric}(V, V)] \, dx_g.$$

One can now repeat the procedure of Method *1*. □

b) Harmonic 1− forms

In Section 3.2, we wrote the Hodge Laplacian only in an abstract form (3.1). As was mentioned at the end of Section 3.3, it can be expressed in a different way as a *Weitzenböck formula* which is often useful:

$$\Delta_H \omega = \nabla^* \nabla \omega + (\text{ curvature})\, \omega, \tag{3.19}$$

where ω is a $p-$ form, ∇ is the covariant derivative, and "curvature" stands for an expression involving the curvature of the manifold (see (**??**). The curvature expression in (3.19) is quite simple in the special case of $1-$ forms when it becomes

$$\Delta_H \omega = \nabla^* \nabla \omega + \mathrm{Ric}\, \omega^\sharp, \tag{3.20}$$

where ω is a $1-$ form, $\omega^\sharp$ is the dual vector field (found using the Riemannian metric g) and Ric is the Ricci curvature of g. If one multiplies (3.20) by ω and integrates by parts, one obtains

$$\langle \omega,\, \Delta_H \omega \rangle = \int [|\nabla \omega|^2 + \mathrm{Ric}(\omega^\sharp,\, \omega^\sharp)].$$

Thus $\nabla \omega = 0$ and $\mathrm{Ric}(\omega^\sharp, \omega^\sharp) = 0$ everywhere. If in addition we have $\mathrm{Ric} > 0$, at one point, then $\omega = 0$. But by Hodge theory we know that the dimension of the space of harmonic $1-$ forms is the first Betti number (see 3.3). Consequently, if a compact manifold has $\beta_1 \neq 0$ then there is no Riemannian metric with positive Ricci curvature. We collect these results in the next theorem.

Theorem 3.3 BOCHNER VANISHING THEOREM *If* (M, g) *is a compact* n-*dimensional manifold with non-negative Ricci curvature, then any harmonic* $1-$ *form has zero covariant derivative, that is, it is parallel. Thus, the first Betti number,* $b_1 \leq n$. *Moreover, if the Ricci curvature at one point is positive, then* $b_1 = 0$.

REMARK 3.1 This proof used Method *2* of the previous theorem. One could also have applied Method *1* as follows. Use the Weitzenböck formula (3.20)to compute $\Delta|\omega|^2$ for any $1-$ form ω and obtain

$$\Delta|\omega|^2 = 2\langle -\Delta_H \omega,\, \omega\rangle + 2|\nabla \omega|^2 + 2\, \mathrm{Ric}(\omega^\sharp,\, \omega^\sharp).$$

This shows that if ω is harmonic and $Ric \geq 0$, then $\Delta|\omega|^2 \geq 0$ so by the maximum principle the right side is zero. One now gets the same conclusion as before.

Using different methods one can prove the stronger assertion that $Ric > 0$ implies the fundamental group, $\pi_1(M)$ is finite. However, the above technique applies in situations where other methods are not available. This technique requires two ingredients:

(i) a Weitzenböck-type formula (3.19)where the "curvature" term is interesting, and

(ii) some topological interpretation of the kernel of the operator.

We will use this procedure again in Section 3.6 when we discuss the Lichnerowicz vanishing theorem. The survey article [Wu] and the book [LM] are good source for more information.

3.5 The Dirac Operator

For the de Rham complex, in Section 3.2.c) above we found a first order self-adjoint elliptic operator D whose square was the Laplacian. The Cauchy-Riemann equations are an even simpler example. This leads one to seek other first order operators which are the square root of the Laplacian.

First we work in $\mathbb{R}^n$ with the Laplacian acting on the vectors $u = (u_1, \ldots, u_N)$. Below we will see that one must pick N appropriately in terms on n. Thus we seek $N \times N$ constant matrices $E_1, \ldots, E_n$ so that

$$\Big(\sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}\Big)^2 = -\Big(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\Big) I. \tag{3.21}$$

We use the minus sign on the right since we want the operator on the right to be the Hodge Laplacian. Expanding the left side, we find that

$$E_j^2 = -I \qquad \text{and} \qquad E_i E_j + E_j E_i = 0 \qquad \text{for } i \neq j. \tag{3.22}$$

Once one has these matrices, the *Dirac operator* is defined by

$$D = \sum E_j \frac{\partial}{\partial x_j} \tag{3.23}$$

and satisfies (3.21).[2] The Dirac operator is a first order self-adjoint elliptic operator. For a given value of n, one must choose N sufficiently large in order to be able to obtain the matrices E_j. Because of the multiplication property 3.22, one can reduce replace any product such as $E_4E_1E_3$ by one where the indices are strictly increasing, $E_1E_3E_4$. Thus, products of the form $E_{j_1}E_{j_2}\cdots E_{j_k}$, where the indices are strictly increasing, $j_1 < j_2 < \ldots < j_k$, form a basis for this algebra of matrices. This basis has 2^n elements. It is a useful exercise in algebra to find the E_j explicitly in the special case when $n = 2$.

A bit more abstractly, matrices $E_1, \ldots, E_n$ with the multiplication rules (3.22)generate an algebra, called the *Clifford algebra*.

One can repeat this replacing R^n by any inner product space V. If $e_1, \ldots, e_n$ are an orthonormal basis, for any vectors u, $v \in V$ the rules (3.22)can be summarized as

$$u \cdot v + v \cdot u = -2\langle u,\, v\rangle. \tag{3.24}$$

Thus, the Clifford algebra $C_n(V)$ of an $n-$ dimensional inner product space V can be described abstractly as the tensor algebra generated by $e_1, \ldots, e_n$ divided out by the ideal defined by (3.24). This also proves that the Clifford algebra does exist. The construction of the exterior algebra of a vector space is quite similar, but it does not use the inner product. From this construction it is clear that the dimension of C_n is 2^n, that is $N^2 = 2^n$; if $n = 2k$ then $N = 2^k$.

The resulting sub-algebra of the algebra of all $N \times N$ matrices give a representation of the Clifford algebra C_n of the inner product space V as matrices acting on a new vector space of dimension N. The $N-$ dimensional space that these matrices act on is called the vector space $\mathcal{S}$ of *spinors*. Thus, spinors are by definition, the vector space on which the Clifford algebra acts. The representation gives a map $\rho : C_n \to \mathrm{End}(\mathcal{S})$. If one works over the complex numbers and if n is even, $n = 2k$, then the algebra is *simple,,* that is, one obtains the whole algebra of $N \times N$ matrices.

[2] Dirac wanted a square root of the wave operation $\Box u = u_{xx} + u_{yy} + u_{zz} - u_{tt}$, but just replace t by iw and this is formally the same as what we did.

REMARK 3.2 In the special case of $V = \mathbb{R}^n$ with $n = 4$, then also $N = 4$ but one should *avoid* the temptation of identifying the 4-dimensional space $\mathcal{S}$ with $\mathbb{R}^4$. A useful exercise is to actually find the matrices $E_1, \ldots, E_4$ for this. There is no "easy" way to go from the vector space V to the related vector space of spinors.

We have carried out this construction for a single inner product space V of dimension $n = 2k$ (this generalizes immediately to a vector space with a non-degenerate quadratic form). For a Riemannian manifold (M, g) of dimension $2k$, it is natural to attempt the same construction at every point, replacing V by the tangent spaces. If one can do this, then one obtains the vector space of spinors at every point, and the manifold is called a *spin manifold*; better, one should say the manifold is *spinable,* in analogy with orientable.

There is no difficulty in doing this over a disk. However, there is a topological obstruction for a manifold to admit a spin structure. One needs to assume M is orientable and that M has a spin structure. The obstruction is expressed using the *Stiefel-Whitney* classes w_1 and w_2. One first needs that the manifold is orientable, that is $w_1 = 0$. To be spinable one also must require that the 2^{nd} Stiefel-Whitney class is zero, $w_2 = 0$.

Using an appropriately adapted connection ∇ on the space of spinors the Dirac operator is written

$$D = \sum_{j=1}^{n} E_j \nabla_j .$$

A good general reference for this material is [LM].

3.6 The Lichnerowicz Vanishing Theorem

Since one has a new elliptic operator, the Dirac operator, one should attempt to see if one can again use the ideas in Bochner's vanishing theorem. The corresponding Weitzenböck-type formula is

$$D^2 = \nabla^* \nabla + \tfrac{1}{4} S \tag{3.25}$$

where S is the scalar curvature of (M, g). By identical reasoning as before, we find that if the scalar curvature of g is positive, then $\ker D^2 = \ker D = 0$ (the elements in $\ker D$ are called *harmonic spinors*).

To use this, we need the analogue of the Betti number β_1; this is supplied by the Atiyah-Singer index theorem. Since $D = D^*$ is self-adjoint, then $i(D^2) = 0$. A nontrivial index can be found by a construction motivated by the example of Section c) above. As our substitute for the spaces Ω^{ev} and Ω^{odd} of differential forms on a manifold of dimension $2k$, let τ be the Clifford product

$$\tau = i^k E_1 E_2 \ldots E_{2k}$$

(essentially the volume element). By an easy computation $\tau^2 = 1$. Since τ is an element of the Clifford algebra, it acts on the spinors $\mathcal{S}$ and has eigenvalues ± 1. Let $\mathcal{S}^+$ and $\mathcal{S}^-$ be the corresponding eigenspaces (one can also define $\mathcal{S}^\pm =$ image of the projector $(1 \pm \tau)/2)$). If $\Gamma(\mathcal{S}\pm)$ is the space of sections of the spinor bundle, then $D : \Gamma(\mathcal{S}^+) \to -(\mathcal{S}^-)$ and $D^- : \Gamma(\mathcal{S}^-) \to -(\mathcal{S}^+)$ so we can define D^+ as the restriction of D to $\Gamma(\mathcal{S}^+)$, and D^- similarly. Then

$$i(D^+) = \dim \ker D^+ - \dim \ker D^- .$$

As a consequence of our observations, if scalar curvature is positive then $i(D^+) = 0$. On the other hand, for an oriented spin manifold of dimension $4k$, the index theorem shows that the index $i(D^+)$ is the $\hat{A}(M)$-genus. Consequently, if $\hat{A}(M) \neq 0$, then M does not admit a metric with positive scalar curvature. This is Lichnerowicz's vanishing theorem. As is discussed in Section 7.2, there are no topological obstructions to negative scalar curvature. Note that if there is no metric with positive scalar curvature, then there is surely no metric with positive Ricci or sectional curvature.

See the discussion in [LM], which also has an enlightening way of obtaining Weitzenböck formulas. Extensions of this and other topological obstructions to scalar curvature will be discussed in Chapter 7.2.

3.7 A Liouville Theorem

As a change, instead of working on a compact manifold, we will let (M, g) be a complete non-compact Riemannian manifold and prove a Liouville-type theorem. The classical version states that a harmonic function in Euclidean n-space that is bounded from below (or above) must be constant. The naive generalization to a complete Riemannian manifold is false, as one can see in hyperbolic space. For instance, if one uses the unit disk in $\mathbb{R}^2$ as the model for hyperbolic space, then the hyperbolic metric with Gauss curvature -4 is

$$g = (dx^2 + dy^2)/(1 - r^2)^2,$$

where $r^2 = x^2 + y^2$. Using (1.6) we find that the Laplace equation in this metric is

$$\Delta u = (1 - r^2)^2 (u_{xx} + u_{yy}) = 0,$$

so *every* function that is harmonic in the Euclidean metric is also harmonic in the hyperbolic metric. In particular, there are many non-constant bounded harmonic functions in the hyperbolic disk.

Yau[Y-1] proved that Liouville's theorem is true for a complete Riemannian metric g if its Ricci curvature is non-negative. In view of the above example, the curvature assumption should not be surprising. He also proved that *for a complete Riemannian manifold, if a function u satisfies $u\Delta u \geq 0$ and if it is in L^p for some $p > 1$ then it must be constant.* Note that no curvature assumption is made. The assunption $u\Delta u \geq 0$ is obviously satisfied both for harmonic functions and for non-negative subharmonic functions (subharmonic means $\Delta u \geq 0$).

We prove this second result. In Euclidean space the desired result (with $u \in L^1$) follows most quickly by letting $R \to \infty$ in the "solid" mean value property:

$$u(x) \leq \frac{1}{\mathrm{Vol}(R)} \int_{|y-x| \leq R} u(y)\, dy$$

where $Vol(R)$ is the volume of the ball of radius R. (This version of the mean value property follows from integrating the usual version — where one has the average only over a sphere — with respect to the radius). Letting $R \to \infty$ one immediately obtains that if $u \in L^1$ then it must be zero. By an application of Hölder's inequality one reaches the same conclusion if $u \in L^p$ for some $p \geq 1$.

For general complete Riemannian manifolds, if one makes the improbable assumption that u has compact support (valid of course in the special case of a compact manifold without boundary), then integrating $u\Delta u \geq 0$ by parts gives

$$0 \leq \int u\Delta u\, dx_g = -\int |\nabla u|^2\, dx_g$$

from which one clearly sees that $\nabla u = 0$ and hence that $u = \text{const}$.

The virtue of this is that it suggests an approach not assuming u has compact support. Introduce a piecewise-linear cut-off function $\eta(t)$ with the properties *1*) $\eta(t) = 1$ for $|t| \leq 1$, *2*) $\eta(t) = 0$ for $t \geq 2$ and 3) $|\eta'(t)| \leq 1$. Fix a point x_0 and consider the balls B_R and B_{2R} centered both centered at x_0 and having radii R and $2R$, respectively. We will use the cut-off function $\varphi(r) = \eta(r/R)$, where r is the Riemannian distance from x_0. Note that because of possible points in the "cut locus" (points where r ceases minimizing the distance because of alternative shorter paths), the function r is not necessarily smooth; however, since $|\nabla r| = 1$ almost everywhere, we see that φ is Lipschitz continuous, which is enough for us.

Multiply $u\Delta u \geq 0$ by φ^2 (not just φ), and integrate by parts over B_{2R} to obtain

$$0 \leq \int_{B_{2R}} \varphi^2 u \Delta u \, dx_g = -\int_{B_{2R}} \left(2\varphi u \nabla\varphi \cdot \nabla u + \varphi^2 |\nabla u|^2\right) dx_g. \tag{3.26}$$

Therefore,

$$\int_{B_{2R}} \varphi^2 |\nabla u|^2 \, dx_g \leq \int_{B_{2R}} 2|\varphi u \nabla\varphi \cdot \nabla u| \, dx_g. \tag{3.27}$$

The elementary — and very useful — inequality $2|xy| \leq cx^2 + c^{-1}y^2$, which is true for *any* $c > 0$, gives the estimate

$$2\varphi u \nabla\varphi \cdot \nabla u \leq c\varphi^2 |\nabla u|^2 + c^{-1} u^2 |\nabla\varphi|^2$$

Applying this with the choice of $c = 1/2$ in the above integral and using the properties of the cut-off function φ we find

$$\begin{aligned} \frac{1}{2}\int_{B_R} |\nabla u|^2 \, dx_g &\leq \frac{1}{2}\int_{B_{2R}} \varphi^2 |\nabla u|^2 \, dx_g \\ &\leq 2\int_{B_{2R}} |\nabla\varphi|^2 u^2 \, dx_g \leq \frac{2}{R^2}\int_{B_{2R}} u^2 \, dx_g. \end{aligned}$$

Hence one can estimate $|\nabla u|$ on one ball in terms of $|u|$ on a larger ball:

$$\int_{B_R} |\nabla u|^2 \, dx_g \leq \frac{4}{R^2}\int_{B_{2R}} u^2 \, dx_g. \tag{3.28}$$

From this, by letting $R \to \infty$ we obtain the Liouville theorem that if $u \in L^2$ and $\Delta u = 0$, then u is constant.

If we know that $u \in L^p$ for some $p > 1$ and $u \geq 0$, we can still obtain the Liouville theorem by a slight modification of this proof. Multiply the inequality $u\Delta u \geq 0$ by $\varphi^2 u^{p-2}$, and integrate by parts over B_{2R} to obtain a replacement for (3.27). Then essentially the same inequalities, only this time using $c = (p-1)/2$, gives the following generalization of (3.28)

$$\int_{B_R} u^{p-2} |\nabla u|^2 \, dx_g \leq \frac{16}{R^2(p-1)^2}\int_{B_{2R}} u^p \, dx_g.$$

Again let $R \to \infty$. This completes the proof.

Note that in the fundamental estimate (3.28) no assumptions were made on the underlying manifold. This inequality also shows that if u is bounded and the volume of the manifold is bounded, then u must be a constant.

3.8 Unique Continuation

a) The Question

Since classical harmonic functions in domains in R^n are real analytic, that is, they have power series expansions, it follows that if a harmonic function has a zero of infinite order at one point of a connected open set, then it must be identically zero in that set. This is the *unique continuation property.* The same property is true for solutions of more general second order elliptic equations whose coefficients are only modestly smooth.

We will consider functions that satisfy

$$|\Delta u| \le a|u| + b|\nabla u|$$

in a open set Ω on an n-dimensional Riemannian manifold (M^n, g). Here Δ is the Laplacian in this metric g

$$\Delta u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\big(g^{ij}\sqrt{g}\frac{\partial z}{\partial x^j}\big),$$

where g^{ij} is the inverse of g_{ij}.

Chapter 4

Nonlinear Elliptic Operators

4.1 Introduction

The lesson one learns from elementary calculus is that the essence of an object is frequently captured by local information embodied in the derivative. Following this, by linearizing a differential equation at a point one can apply the theory of linear differential operators to nonlinear ones. The sort of information one obtains for the equation is *despite* its being nonlinear. Smoothness of solutions is a typical local result that carries over to nonlinear equations.

4.2 Differential Operators

In local coordinates on $\mathbb{R}^n$ one frequently finds differential operators of the special form

$$F(x, \partial^k u) = \sum_{|\alpha|=k} a^\alpha(x, \partial^\ell u)\partial^\alpha u + f(x, \partial^\ell u), \tag{4.1}$$

where u is a vector-valued functions, α is a multi-index and $|\ell| < k$. Such an equation is called *quasilinear* since it is linear in the highest derivatives of u. It is genuinely *linear* if in addition the coefficients a^α —which may be complex matrices—do not depend on u or its derivatives and f is linear in $\partial^\ell u$. The usual formulas for the Riemann and Ricci curvatures of a Riemannian manifold can be viewed as examples of second order quasilinear differential operators in the metric g. Other examples are the minimal surface equation—as well as all Euler-Lagrange equations for problems in the variations. On the other hand, the Gauss curvature, K, of a surface $z = u(x, y)$ in $\mathbb{R}^3$ satisfies

$$u_{xx}u_{yy} - u_{xy}^2 = K(x, y)(1 + u_x^2 + u_y^2)^2, \tag{4.2}$$

which is not quasilinear, it is fully nonlinear. Another fully nonlinear equation is the Monge-Ampère equation for Kähler-Einstein metrics. It is customary to refer to nonlinear equations involving the determinant of the hessian as "Monge-Ampère equations". We will discuss Kähler-Einstein metrics in Chapter 9.3.

One frequently meets second order equations, both linear and quasilinear, in *divergence form*

$$\operatorname{div} A(x, u, Du) = f(x, u, Du), \tag{4.3}$$

where A is a vector field. The model case is when $A = \text{grad } u$

$$\Delta u = f(x, u, Du).$$

For example, on $\mathbb{R}^n$ equation (4.3) is

$$\sum_i \frac{\partial}{\partial x^i} A^i(x,u,Du) = f(x,u,Du).$$

If $u \in C^2$ is a solution of (4.3) in an open set $\Omega \subset \mathbb{R}$, then for any $\varphi \in H^{2,1}$ with support in Ω, if we multiply (4.3) by φ and integrate by parts we have

$$-\int_\Omega \sum_i A^i(x,u,Du)\frac{\partial \varphi}{\partial x^i}\,dx = \int_\Omega f(x,u,Du)\varphi\,dx. \tag{4.4}$$

Conversely, if some $u \in C^2$ satisfies this integral identity for all smooth φ with support in Ω, then since these φ are dense in $C(\Omega)$ one can reverse these steps and deduce that u is in fact a solution of (4.3). The point, however, is that the integral identity (4.4) only involves the *first* derivatives of u; in many circumstances (4.4) makes sense even for $u \in H^{2,1}(\Omega)$ rather than $C^2(\Omega)$ (as in the model above where $A(x,u,Du) = \text{grad } u$). We say that $u \in H^{2,1}(\Omega)$ is a *weak solution* of the original equation if the integral identity (4.4) holds for all $\varphi \in H^{2,1}$ with support in Ω. Caution: there are several other useful, but *inequivalent*, definitions of "weak solution".

The main application of these notions is in the *calculus of variations*, where one seeks a critical point of a functional

$$J(u) = \int_M F(x,u,Du)\,dx_g.$$

Then the Euler-Lagrange equation is automatically in divergence form so one may think of (4.4) as a "weak form" of the Euler-Lagrange equation. We already have seen a special case of this in our discussion of the eigenvalues of the Laplacian in Chapter 3.3.

The virtue of enlarging the class of admissible solutions to allow weak solutions is that this may make it much easier to prove the existence of a solution of the equation. On the other hand, one is then faced with the often difficult regularity problem of determining to what extent this weak solution is actually smooth.

4.3 Ellipticity

For a nonlinear differential operator $F(x, \partial^k u)$, its *linearization* or *first variation* at u is the linear operator

$$Lv = \frac{d}{dt}F(x, \partial^k(u+tv))|_{t=0}. \tag{4.5}$$

Thus, the quasilinear operator

$$F(x,\partial^k u) = \sum_{|\alpha|=k} a^\alpha(x,\partial^\ell u)\partial^\alpha u + f(x,\partial^\ell u), \tag{4.6}$$

(recall $\ell \le k-1$) has as its linearization at u the k-th order linear operator

$$Lv = \sum_{|\alpha|=k} a^\alpha(x,\partial^\ell u)\partial^\alpha v + \text{ lower order terms}, \tag{4.7}$$

while the linearization of the Gauss curvature formula (4.2) at u is

$$Lv = u_{yy}v_{xx} - 2u_{xy}v_{xy} + u_{xx}v_{yy} + \text{ lower order terms}. \tag{4.8}$$

The *nonlinear equation* $F(x,\, \partial^k u) = 0$ is said to be *elliptic at* (x,u) (that is, it is elliptic at x for the function u), if its linearization at u is elliptic at the point x. As in Section 2.2 there is an obvious definition for *underdetermined* and *overdetermined elliptic*.

EXAMPLE 4.1 The linearization of $yu_{xx} + uu_{yy} = 0$ at u is

$$Lv = yv_{xx} + uv_{yy} + \text{ lower order terms.}$$

This is elliptic at the points where both y and $u(x, y)$ have the same sign.

EXAMPLE 4.2 The following formula gives the *mean curvature* H of a graph $w = u(x)$ in $\mathbb{R}^{n+1}$

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = H, \tag{4.9}$$

When viewed as a differential equation for u. It is straightforward to verify this is elliptic for all functions u at all points. *Minimal surfaces* are the special case $H = 0$.

EXAMPLE 4.3 In studying *two dimensional irrotational steady fluid flow* one is led to

$$(c^2 - \varphi_x^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (c^2 - \varphi_y^2)\varphi_{yy} = 0.$$

Here $V = \operatorname{grad} \varphi = (\varphi_x, \varphi_y)$ is the velocity vector of the fluid and c is the speed of sound in the fluid. This equation is elliptic at a point if $|\operatorname{grad} \varphi| < c$ there, that is the speed of the flow $|V|$ is less than the speed of sound. If $|V| > c$ the flow is supersonic and the equation is hyperbolic. One can then have shock waves which are quite different than the smoothness associated with solutions in the subsonic case in which the equation is elliptic.

EXAMPLE 4.4 The *Monge-Ampère* equation

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y) \tag{4.10}$$

is elliptic at a solution $u(x, y)$ precisely at the points where $f(x, y) > 0$. Similarly, from (4.8) it is clear that the Gauss curvature equation (4.2) is elliptic precisely at those points where $K > 0$.

4.4 Nonlinear Elliptic Equations: Regularity

Fully nonlinear elliptic equations and systems

$$F(x, \partial^k u) = 0. \tag{4.11}$$

enjoy many of the same local regularity results as do linear systems. Notice that if we take the partial derivative $\partial/\partial x^i$ of (4.11) then $\partial u/\partial x^i$ satisfies a quasilinear equation so the results in Section 2.4 apply.

Theorem 4.1 ELLIPTIC REGULARITY *Assume that for some integer* $\ell \geq 1$, $0 < \sigma < 1$

$$F(x, s) \quad \text{is in} \quad C^1 \quad \text{or} \quad C^{\ell,\sigma} \quad \text{or} \quad C^\infty \quad \text{or} \quad C^\omega$$

as a function of all its variables for x *in an open set* $\Omega \subset \mathbf{R}^n$ *and all* s, *and that* $u \in C^k(\Omega)$ *is an elliptic solution of equation* (4.11). *Then for any* $0 < \lambda < 1$

$$u(x) \quad \text{is in} \quad C^{k,\lambda} \quad \text{or} \quad C^{k+\ell,\sigma} \quad \text{or} \quad C^\infty \quad \text{or} \quad C^\omega$$

respectively, in Ω.

See [ADN-2, Theorem 12.1] and [Mo, Theorems 6.7.6 and 6.8.1] for a proof. The key ingredient is Theorem 2.3 above. Regularity for overdetermined elliptic systems near the point x_0 is obtained by considering the elliptic system $L_0^* F(x, \partial^\ell u) = 0$, where L_0 is the linearization of F at $(x_0, j_k u)$ with $j_k u = k$-jet of u at x_0. There are also important recent results. See [C-1] and [C-2] for additional information and references.

EXAMPLE 4.5 An immediate but striking example is that any C^2 surface with constant mean curvature H must be real analytic, since it satisfies the elliptic equation (4.9) which has analytic coefficients. In particular, this is true for the special case $H = 0$ of minimal surfaces. By working harder, one can weaken the initial assumption that the surface is C^2.

Similarly, a piece of a surface with constant positive Gauss curvature must be real analytic because the equation (4.2) is analytic, and is elliptic if the curvature is positive. □

4.5 Nonlinear Elliptic Equations: Existence

There is no general existence theory for nonlinear equations; indeed, we know little in general about solving simultaneous nonlinear equations in finite dimensional Euclidean space. For nonlinear partial differential equations the subject essentially consists of some significant examples and several techniques that have been useful. All the techniques are direct generalizations of those used in finite dimensional case. In this section we will limit ourselves to stating two results, both of which are consequences of the implicit function theorem, and then giving a list of some other methods. The remaining chapters of this volume is a collection of examples of these methods.

The first result (from [ADN-2, §12]) considers the question of solving a nonlinear equation (or system) of order k

$$F(x, \partial^k u; t) = 0, \tag{4.12}$$

where t is a real parameter. Say one has a solution at $t = t_0$. Can one always find a solution for t near t_0? It is reasonable that the implicit function theorem (in Banach spaces) is the key to this. Recall that we are on a compact manifold M without boundary so the question of boundary conditions does not enter.

Theorem 4.2 [PERTURBATION] *In* (4.12)*, let* F *be a* C^∞ *function of all of its arguments. Assume that*

(i) $u_0 \in C^k$ *is a solution of* (4.12) *for* $t = t_0$,

(ii) *the linearization,* L, *at* u_0 *is elliptic, and*

(iii) *the linearized equation* $Lv = f$ *has a unique solution for any* $f \in C^\sigma$ *(for some* $0 < \sigma < 1$*).*

Then there is a solution $u \in C^{k,\alpha}$ *of* (4.12) *if* $|t - t_0|$ *is sufficiently small.*

Proof. By Theorem (4.1), we know that $u_0 \in C^{k,\alpha}(M)$, indeed $u_0 \in C^\infty(M)$. Let $T(u,t) = F(x, \partial^k u, t)$, so $T : C^{k,\alpha}(M) \times R \to C^\alpha(M)$ is a smooth map. Note that $T(u_0, t_0) = 0$, while $T_u(u_0, t_0) = L : C^{k,\alpha}(M) \to C^\alpha(M)$ is bijective. The desired conclusion now follows from the standard implicit function theorem in Banach Spaces. Because we assumed $F \in C^\infty$, then we also know that $u \in C^\infty$, but it is obvious that this proof requires only very mild smoothness of F if we only want to obtain a solution $u \in C^{k,\alpha}$. □

This theorem is often used in the "continuity method" (see below and Chapter 5) as well as in a variety of perturbation situations. If the linearization, L, is elliptic but *not* invertible, one can investigate the higher order terms in the Taylor series of $T(u,t)$ near (u_0, t_0) and (attempt to) determine when the nonlinear equation $T(u,t) = 0$ is solvable.

This is called "bifurcation theory" (see Chapter 6.7 for a brief discussion and [N-3] for a more thorough treatment).

Several straightforward modifications of this perturbation Theorem 4.2 are often required in practice.

(1) In the frequently occurring case when (4.12) is quasilinear, one can use Sobolev spaces instead of Hölder spaces. One uses $L: H^{p,k} \to L^p$ where $p > \dim M$, since then by the Sobolev embedding theorem the coefficients in (4.6) with $\ell \le k-1$ are continuous. An example is [KW-2, 3] and Chapter 6.4 below.

(2) Underdetermined elliptic systems can often be treated by combining the perturbation theorem with the device in the proof of part $a)$ in Corollary 2.5 (see Chapter 6.3 and 6.4).

(3) There is a simple situation where the perturbation theorem can be used with L not invertible. This is when L is invertible on a subspace and this subspace is "invariant" under F. For example, if $A = \{f \in C^0(M) : \int_M f\, dx_g = 0\}$, and if $F: C^{k,\alpha} \cap A \to C^\alpha \cap A$ with $L: C^{k,\alpha} \cap A \to C^\alpha \cap A$ invertible (which is the case if $L = \Delta$), then the Perturbation Theorem applies to yield a solution $u \in C^{k,\alpha} \cap A$ (presuming $u_0 \in A$ too). See [Au-1], [Y-2], and Chapter 9.3.b) where this occurs in obtaining a Kähler-Einstein metric.

One rather simple-sounding question is if one can find some solution of a nonlinear equation (or system) of order k,

$$F(x, \partial^j u) = 0, \qquad |j| \le k$$

in a neighborhood of a point $x \in \mathbb{R}^n$. The question is quite modest, since we seek a solution only in some neighborhood of a point x_0, not on a compact manifold and do not impose any boundary conditions. To have some perspective, we point out that the deceptively simple-looking linear equation in $\mathbb{R}^2$

$$u_x + ixu_y = f(x, y) \tag{4.13}$$

has *no* solution in any neighborhood of $x = 0$ for most $f \in C^\infty$. This celebrated surprising fact was first found by H. Lewy in 1956 (he gave a slightly different example). If f is analytic, of course one always can use power series to find a solution. If one prefers equations with real coefficients, one can take the real and imaginary parts of (4.13) to get a pair of real equations.

However for $F(x, s)$ smooth, if one assumes ellipticity there is no difficulty locally solving

$$F(x, \partial^j u) = 0, \qquad |j| \le k \tag{4.14}$$

as long as the equation is solvable at one point (to avoid silly unsolvable examples such as finding a real solution of $(\Delta u)^2 = -1$).

Theorem 4.3 [LOCAL SOLVABILITY] *Assume $F(x, s)$, is a C^∞ function of all of its arguments and that the function $u_0(x) \in C^k$ is an elliptic solution at $x = x_0$. Then in some neighborhood of x_0 there is a solution $u \in C^k$ (and hence C^∞) of (4.14). Moreover, u is near u_0 and one can also specify that $\partial^\alpha u = \partial^\alpha u_0$ for $|\alpha| \le k-1$ at x_0.*

The proof of this uses the standard implicit function theorem in Banach spaces. By a preliminary change of variables, one may assume that $x_0 = 0$ and $u_0 \equiv 0$, so the solvability at $x_0 = 0$ means $F(0,0) = 0$. Make the change of scale $x = \lambda y$, $u = \lambda^k v$. Then (4.14) for $v(y)$ becomes (with $\partial_y = \partial/\partial y$)

$$T(v;\lambda) := F(\lambda y,\, \lambda^{k-|j|}\,\partial_y^j v) = 0, \qquad |j| \le k.$$

It is enough to find some $\lambda > 0$ so that we can solve this in the ball $|y| < 1$. Clearly $T(0;0) = 0$. To apply the implicit function theorem we need that the linearization $T_v(0;0)$ is invertible as a map between appropriate Banach spaces. Standard machinery for linear elliptic equations with constant coefficient allows one to complete the proof. [It is instructive first to carry out the details for the *ordinary* differential equation $u' = f(x,u)$ with $u(0) = 0$. Here $T : C^1 \times \mathbb{R} \to C^0$ (instead of C^1, it is more convenient to use the subspace of $u \in C^1$ with $u(0) = 0$.] See [Ma, Section 9], and [N-2, pp. 15-16], for a detailed proof. □

One can call u_0 an "infinitesimal solution" in which case the theorem states that *if $F(x, \partial^k u)$ is elliptic at u_0, then infinitesimal solvability implies local solvability*. Again, using the device of part *a*) in the proof of Corollary 2.5(b) one can prove the local solvability of underdetermined elliptic equations; one application is in Chapter 6.5, another is Malgrange's proof of the Newlander-Nirenberg theorem (see Section 6.3 below or [Ma] or [N-2]).

A different local solvability theorem is true if the linearization of F at (x_0, u_0) is strongly elliptic; then one can solve the Dirichlet problem in a small disc, instead of asking that $\partial^\alpha u = \partial^\alpha u_0$, $|\alpha| \le k-1$, at the origin.

EXAMPLE 4.6 MONGE-AMPÈRE, LOCALLY An immediate application is the local solvability of $u_{xx}u_{yy} - u_{xy}^2 = f$ near the origin if $f(0,0) = c^2 > 0$, since $u_0 = c(x^2 + y^2)/4$ is an elliptic solution at the origin.

It is also locally solvable if $f(0,0) < 0$ by using techniques from the theory of hyperbolic equations. However, if $f(0,0) = 0$, then—even for $f \in C^\infty$—we do not yet know if one can always locally find a solution $u \in C^2$ of this equation. If f is real analytic, then it is locally solvable since one can find a power series solution (Cauchy-Kowalewskaya theorem). If either $f(x,y) \ge 0$ near the origin, or $\nabla f(0,0) \ne 0$ then the locally solvability was proved by C-S Lin [Lin-1], [Lin-2]. □

4.6 A Comparison Theorem

This section has elementary comparison results for a second order scalar operator $F(x, u, Du, D^2u)$, where, in local coordinates, $F(x,\, s,\, p_i,\, r_{ij})$ is a C^1 function of its variables with the matrix $(\partial F/\partial r_{ij})$ positive definite. The operator F is then elliptic for all u and for all x in a domain Ω, which could be a manifold with or without boundary.

We begin with a routine procedure for applying theorems for linear problems to a nonlinear problem. Say $u(x), v(x) \in C^2(\Omega)$. Let $w = u - v$ and $z(x;t) = v(x) + t[u(x) - v(x)]$. Then

$$\begin{aligned} F(x,\, u,\, Du,\, D^2u) &- F(x,\, v,\, Dv,\, D^2v) \\ &= \int_0^1 \frac{\partial}{\partial t} F(x,\, z(x;t),\, Dz(x;t),\, D^2 z(x;t))\, dt \qquad (4.15) \\ &= \sum_{i,j} a^{ij}(x)\, \frac{\partial^2 w}{\partial x^i \partial x^j} + \sum_i b^i(x)\, \frac{\partial w}{\partial x^i} + c(x)\, w, \qquad (4.16) \end{aligned}$$

where the matrix

$$a^{ij}(x) = \int_0^1 \frac{\partial F}{\partial r_{ij}}(x,\, z(x;t),\, Dz(x;t),\, D^2 z(x;t))\, dt$$

is positive definite, and $b^i(x)$, $c(x)$ are given by similar formulas. The operator on $w(x)$ in equation (4.16) is *linear* elliptic. Observe that if $\partial F/\partial s \le 0$ then $c(x) \le 0$ so one can apply the strong maximum principle. For nonlinear systems of equations many of these same ideas still apply.

It is occasionally useful to note that $z(x;t)$ could have been any path of functions from $v(x) = z(x;0)$ to $u(x) = z(x;1)$ and that the conditions, such as ellipticity, need hold only for these function $z(x;t)$.

With $F(x,s,p,r)$ as above, the strong maximum principle of Section 2.6 now immediately implies the following comparison theorem and its corollary.

Theorem 4.4 [COMPARISON THEOREM] *Let Ω be either a bounded domain with boundary or a compact manifold without boundary. Assume that $u,\, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$F(x,\, u,\, Du,\, D^2 u) \ge F(x,\, v,\, Dv,\, D^2 v) \tag{4.17}$$

in Ω and that the matrix $(\partial F/\partial r_{ij})$ and also $\partial F/\partial s \le 0$.
a) *If Ω has a boundary and $u \le v$ on the boundary, then $u \le v$ thoughout Ω. If $u(x) = v(x)$ at some interior point, then $u \equiv v$.*
b) *If Ω is a compact manifold without boundary, then u and v differ at most by a constant, while if $\partial F/\partial s < 0$ then $u \equiv v$.*

Corollary 4.5 [UNIQUENESS] *If in part* a) *above we assume both*

$$F(x,\, u,\, Du,\, D^2 u) = F(x,\, v,\, Dv,\, D^2 v) \qquad in \quad \Omega$$

and $u = v$ on the boundary, then $u \equiv v$ throughout Ω.

A simple geometric example illustrates these ideas.

EXAMPLE 4.7 [MEAN CURVATURE] Let $u(x)$ and $v(x)$ be graphs of hypersurfaces having constant mean curvature H for x in a connected open set $\Omega \subset \mathbb{R}^n$. If $u(x) \le v(x)$ then we claim that either $u(x) < v(x)$ or else the surfaces are identical. In otherwords, if they are tangent at one point then they coincide. The proof is an immediate application of the above comparison theorem to the mean curvature equation (4.9).

Identical reasoning verifies some geometric intuition arising from the special case of tangent hemispheres of different radii: if they are oriented so they are both concave (or both convex), then the hemisphere with the larger curvature must be inside the other hemisphere. More generally, if the surfaces $u(x) \le v(x)$ have different constant non-negative mean curvatures H_1 and H_2, respectively, and if these surfaces are tangent at one point, then the only way they can be distinct is if $H_2 < H_1$, the surface with the larger mean curvature being "inside" the one with the smaller curvature.

4.7 Nonlinear Parabolic Equations

Just as in the linear case, many of the circle of ideas for nonlinear elliptic equations also apply to nonlinear parabolic equations. We will only need second order equations of the form

$$\frac{\partial u}{\partial t} = F(x, t, \partial_x^j u), \qquad \text{where} \quad |j| \le 2. \tag{4.18}$$

We assume that F is a smooth function of its variables. As one might anticipate, this nonlinear system is called *parabolic* at a function u if its linearization at u is parabolic in the sense that it has the form (2.27).

The only significant result we need is an existence theorem for the initial value problem for some nonlinear parabolic equations. From the example $du/dt = u^2$, $u(0) = a$, whose solution is $u(t) = 1/(c-t)^2$, the most we can hope for is existence for some short interval of time, $0 \le t \le \epsilon$. We consider the initial value problem of solving equation (4.18) in $M \times [0, \epsilon]$, with $u(x, 0) = \varphi(x)$, where, say, $\varphi \in C^\infty(M)$.

Theorem 4.6 [Short time existence]. *Under the above assumptions, there is an* $\epsilon > 0$ *so that the problem* (4.18) *has a unique solution* $u(x,t)$ *for* $x \in M$, $0 \le t \le \epsilon$. *Moreover,* $u \in C^\infty(M \times [0, \epsilon])$.

One can either prove this directly from Theorem 2.15, using iterations, or else use the Implicit Function Theorem (see [H-1, p. 122-123] for an example). As an alternate, at least for quasilinear equations, one can use the Schauder Fixed Point Theorem (see [F, p. 177-181] or [LSU, p. 596]).

A glance at the proof of the Comparison Theorem 4.4 above shows that it has a straightforward extension to the initial value problem for a nonlinear scalar parabolic equation (4.18).

4.8 A List of Techniques

Finally, we list various methods that have been used to prove existence for nonlinear elliptic problems. Many of these methods will be discussed in greater detail in subsequent chapters, particularly Chapter 5 which contains short illustrative examples. See also the survey article [N-4].

Continuity Method. To solve the second order equation $F(x, \partial^2 u) = 0$, you consider a family of problems

$$F(x, \partial^2 u; t) = 0, \quad \text{for} \quad 0 \le t \le 1,$$

where $F(x, \partial^2 u; 1) = F(x, \partial^2 u)$ is the problem you wish to solve, while $F(x, \partial^2 u; 0) = 0$ is a simpler problem that you know how to solve. Let A be the set of $t \in [0,1]$ such that the problem is solvable. By construction $t = 0 \in A$. One shows that A is open, usually by the Perturbation Theorem 4.2. The final step is to show A is closed, since then $A = [0,1]$. Say $A \ni t_j \to \tau$. Then there are u_j satisfying $F(x, \partial^2 u_j; t_j) = 0$. To prove that $\tau \in A$ (that is, A is closed) one tries to find a subsequence of the $\{u_j\}$ that converges in C^2. The standard approach is to show that the sequence u_j is in a bounded set in $C^{2,\sigma}$ for some $0 < \sigma < 1$ and then apply the Arzela-Ascoli theorem. Thus, one has the (possibly difficult) task of finding an *a priori estimate*: if u is a solution $F(x, \partial^2 u; t) = 0$ for some $0 \le t \le 1$, then $\|u\|_{2+\alpha} \le$ *constant*, where the constant is independent of t. Two applications of this method are in [N-1], and [Au-1], [Y] (see also [SP] and Chapters 3.2 and 6.3 below).

Calculus of Variations. One proves that an equation has a solution by showing that it is the Euler-Lagrange equation of a variational problem (and hence quasilinear), and then proving that this variational problem has a critical point. See [C], [KW-1], [L], [SU], and Chapter 5 below.

Schauder Fixed Point Theorem. One proves that $F(x, \partial^2 u) = 0$ has a solution by showing that a related equation involving a compact operator has a fixed point (this is an extension of the Brouwer Fixed Point Theorem). For example, to prove the existence of a solution of a second order quasilinear equation one might proceed as follows. Given a function v, let $u = T(v)$ be the solution of the linear problem (so we are assuming *linear* solvability)

$$\sum_{|\alpha|\le 2} a^\alpha(x, \partial^\ell v)\partial^\alpha u = f(x, \partial^\ell v), \quad \text{where} \quad \ell \le 1. \tag{4.19}$$

A fixed point $u = T(u)$ is then a solution of the corresponding quasilinear equation. Note that $T : C^{1,\sigma} \to C^{2,\sigma} \hookrightarrow C^{1,\sigma}$ is a compact operator since the inclusion $C^{2,\sigma} \hookrightarrow C^{1,\sigma}$ is compact. To apply the method, one approach is to find a ball $B = \{u : \|u\| \le c\}$ in $C^{1,\sigma}$ and show that $T : B \to B$; thus, one wants to show that if $\|v\| \le c$ then $\|u\| = \|T(v)\| \le c$, i.e., find an *a priori bound* on solutions of $T(v) = u$. Observe that in (4.19) one can put some of the terms involving only *first* derivatives on either side of the equation. This flexibility is frequently exploited. For examples and some modified versions, see Chapter 5.4 below, as well as [CH], [GT, Chapter 10], and [N-3].

Leray-Schauder Degree. This is similar to (but more complicated than) the fixed point approach. It is an extension of the Brouwer degree to Banach spaces. See Chapter 5.4 and [N-3].

Sub and Supersolutions. While only applying to second order scalar equations, this method is often quite simple—when it works. A function u_- is called a *subsolution* of $-\Delta u = f(x, u)$ if $-\Delta u_- \le f(x, u_-)$, with the inequality reversed for a *supersolution* u_+. If there are $u_\pm$ with $u_- \le u_+$, then there is a solution u with $u_- \le u \le u_+$. As an easy simple illustration, one can use constants for $u_\pm$ in the equation $\Delta u = -1 + f(x)e^u$, assuming that $f > 0$, thus proving the existence of a solution (which is unique, since if $w = u - v$, where u and v are solutions, then $\Delta w = -c(x)w$ for some function $c > 0$; hence $w = 0$ by our discussion of (2.13). More complicated cases are discussed in Chapter 5.5 and Chapter 7.

Monotonicity. This method applies to some quasilinear equations that do not quite fit in the calculus of variations approach. See [Mo, § 5.12].

Heat Equation. One solves the "heat equation" $\partial u/\partial t = F(x, \partial^k u)$ and shows that as $t \to \infty$ the solution approaches "equilibrium." Then $\partial u/\partial t \to 0$, and in the limit one obtains a solution of $F(x, \partial^k u) = 0$. A notable application [ES] is to prove the existence of harmonic maps (see also [EL]). Another application is the recent result [H-2], which is also discussed in Chapter 9.2. A simple illustrative example of the method is given in Chapter 5.6.

Alexandrov's Method. This applies only to second order scalar equations $F(x, \partial^\alpha u) = 0$, $|\alpha| \le 2$, where u is a convex function. Use the convexity to obtain approximate polyhedral solutions, and then pass to the limit. A significant application was to give one of the existence proofs for the Minkowski Problem. See [P-1, 2] and [CY].

Steepest Descent. This is an alternate to the calculus of variations. To minimize a functional—and hence obtain a solution of the corresponding problem in the calculus of variations—one follows the gradient lines. There is a close resemblance to the heat equation procedure, see [I] for an example.

Chapter 5

Examples of Techniques

5.1 Introduction

In many ways, partial differential equations is a subject whose essence is more a body of techniques rather than a body of theorems. One of the easiest way to learn these techniques is to see how they can be applied to simple examples. For simplicity, assume $a, h \in C^\infty(M)$. Throughout this chapter we will assume there is some Riemannian metric g prescribed on an n-dimensional compact manifold M without boundary, that ∇ is the gradient and Δ the associated Laplacian.

We shall use a variety of techniques to give many proofs that if $a(x) > 0$ and $h(x) > 0$ are given functions and $\alpha > 1$ is a constant, then the equations

$$\Delta u + a - he^u = 0 \tag{5.1}$$

and

$$\Delta v + av - hv^\alpha = 0, \quad v > 0 \tag{5.2}$$

have unique solutions (unique positive solution in the case of (5.2)).

First, with no sign assumption on $a(x)$ one can always reduce (5.1) to the case where a is a constant by letting $\Delta z = \overline{a} - a$, with $\overline{a} = (\mathrm{Vol}(M)^{-1} \int_M a\, dx_g$ (since $\int_M (a - \overline{a}) dx_g = 0$, there is a solution z, and it is unique up to a constant). Write $u = v + z$. Then (5.1) reduces to solving

$$\Delta v + \overline{a} - He^v = 0, \tag{5.3}$$

with $H(x) = h(x)e^{z(x)}$ a known function. Upon integrating (5.1) over M we obtain $\int_M he^u\, dx_g = \overline{a}\, \mathrm{Vol}(M)$, so that a necessary condition to be able to solve (5.1) is that in some open set $h(x)$ has the same sign as $\overline{a}$ (if $\overline{a} = 0$, then this condition is that h changes sign, unless $h \equiv 0$). Below, we shall observe that in a geometry problem, this necessary sign condition is related to the Gauss-Bonnet theorem. Section 5.7 contains a summary for (5.1), including information on cases we have not treated.

These equations arise in geometry in the following way. Let g be a given Riemannian metric on M^n and let g_1 be a metric *pointwise conformal* to g, so we may write

$$g_1 = e^{2u} g.$$

If S and S_1 are the scalar curvatures of g and g_1, respectively, then from (A.38)

$$2(n-1)\Delta u + (n-1)(n-2)|\nabla u|^2 = S - S_1 e^{2u}. \tag{5.4}$$

For $n = 2$ it is customary to use the Gauss curvature $K = \frac{1}{2}S$ and rewrite (5.4) as

$$\Delta u = K - K_1 e^{2u} \tag{5.5}$$

If we integrate (5.5) over the surface M (writing dA for the element of area) by Gauss-Bonnet we get

$$\int_M K_1 e^{2u}\, dA = \int_M K\, dA = 2\pi\chi(M),$$

which is hardly a surprise since $dA_1 = e^{2u} dA$. Comparing (5.5) with (5.1) we observe that $a = -2K$, $h = -2K_1$, and u is replaced by $2u$. Thus, in view of the reduction above, the case we are considering in this chapter is when $\overline{K} < 0$, that is, when the Euler characteristic is *negative*. Existence of a solution of this equation in the particular instance when $K_1 = -1$ implies that there is a conformal metric with constant negative Gauss curvature, a fact usually associated with the uniformization theorem.

For $n \neq 2$, one can make a change of variable to eliminate the terms involving $|\nabla v|^2$: try the general substitution $v = F(u)$ in (5.4), and then chooses $F(u)$ to eliminate the $|\nabla v|^2$ terms. This leads to the change of variable $v = e^{bu} > 0$, where $b = (n-2)/2$; and results in the simpler appearing equation

$$-\frac{4(n-1)}{n-2}\Delta v + Sv = S_1 v^{(n+2)/(n-2)}, \tag{5.6}$$

that we used in (5.2). In terms of v, we have

$$g_1 = v^{4/(n-2)} g.$$

Part *b)* of the Comparison Theorem 4.4 shows there is at most one solution of (5.1) if $h > 0$, as well as (5.4) if $S_1 < 0$. Motivated by (5.4)-(5.6), if one first makes the change of variable $v = e^w > 0$ then uniqueness of the positive solution of (5.2) follows from the observation that the corresponding equation for w has a unique solution if $h > 0$.

In what follows, all of the results can be generalized; however here our goal is simplicity, not generality. We repeat that that in geometric applications, $a > 0$ and $h > 0$ in (5.1)-(5.2) are the *negative* curvature cases. To get some feeling for differential equations such as (5.1) and (5.2), it is often very helpful to consider first the case when M is a compact *one* dimensional manifold, namely S^1; then, for instance (5.1) becomes the ordinary differential equation $u'' + a = he^u$, that presents fewer technical — and psychological — difficulties, yet is still not trivial.

We first reduce (5.1) and (5.2) to an equation with bounded nonlinearity by the following device. Let $g(x,s) : M \times \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that there exist numbers $s_- < s_+$ so that

$$\text{if } s > s_+ \text{ then } g(x,s) > 0; \qquad \text{if } s < s_- \text{ then } g(x,s) < 0 \tag{5.7}$$

and consider the equation

$$\Delta u = g(x,u). \tag{5.8}$$

Note that both (5.1) and (5.2) have the form (5.8), with the condition (5.7) holding. In addition, for (5.2) we can choose $s_- > 0$, which will be important to insure that the solution obtained by applying this procedure is positive.

Observe that at a maximum of u one has $g(x,u) = \Delta u \le 0$ so $u \le s_+$. Similarly $u \ge s_-$ at a minimum of u. Thus any solution of (5.8) satisfies the *a priori* inequality

$s_- \le u \le s_+$. To obtain an equation with a bounded nonlinearity we will modify $g(x,s)$ for $s < s_-$ and $s > s_+$.

Let $\eta(s) \in C^\infty(\mathbb{R})$ satisfy

$$\eta(s) = \begin{cases} \eta(s_- - 1), & s \le s_- - 1 \\ s, & s_- \le s \le s_+ \\ \eta(s_+ + 1), & s \ge s_+ + 1 \end{cases}$$

as well as $0 \le \eta'(s) \le 1$, and let $f(x,s)$ be the bounded function

$$f(x,s) = g(x,\eta(s)) - \eta(s). \tag{5.9}$$

Consider the equation

$$\Delta u - u = f(x,u). \tag{5.10}$$

If u is a solution of (5.8), then we just proved that $s_- \le u \le s_+$ so u is also a solution of (5.10). The converse is also true.

Lemma 5.1 *Assume* g *satisfies* (5.7) *and define* f *by* (5.9)*). Then a solution* u *of* (5.8) *or* (5.10) *has the property* $s_- \le u \le s_+$. *Consequently* u *satisfies* (5.8) *if and only if it satisfies* (5.10)*.*

Proof. All that remains to be proved is that if u satisfies (5.10) then $s_- \le u \le s_+$. Consider the point $x_{\max}$ where u has its maximum. If $\max u < s_-$ then clearly $u \le s_+$ so consider the case where $\max u \ge s_-$. Then at $x_{\max}$ we know $u \ge \eta(u)$ so that

$$0 \ge \Delta u = u + f(x,u) \ge \eta(u) + f(x,u) = g(x,\eta(u)).$$

By (5.7) $\eta(u) \le s_+$ and hence $u \le s_+$. Thus $\max u \le s_+$. Similarly $\min u \ge s_-$. □

As mentioned above, for (5.2) we have $s_- > 0$ that will insure that the solution of the corresponding equation (5.10)) will be positive.

REMARK 5.1 Equations of the form $\Delta u = f(x,u)$ can have a continuum of solutions. The obvious case is when $f(x,s) \equiv 0$, so any constant is a solution. One can modify this to find other examples, say with all possible solutions lying in a bounded set. One example is $-\Delta u + u = f(u)$, where $f(s) = s$ for $|s| \le 1$ and $|f(s)| \le 2$ everywhere. Then $u(c) = c$ is a solution for every constant $|c| \le 1$, while by the maximum principle all solutions lie in the bounded set $|u(x)| \le 2$.

5.2 Calculus of Variations

We will now use the calculus of variations to solve (5.10). The special case of finding the eigenvalues of the Laplacian, which we treated in Section 3.3 will serve as a useful model. Let

$$F(x,s) = \int_0^s f(x,t)\,dt, \tag{5.11}$$

so $F_s(x,s) = f(x,s)$, and define the functional J by

$$J(u) = \int_M (|\nabla u|^2 + u^2 + 2F(x,u))\,dx_g. \tag{5.12}$$

It is straightforward to verify that $\Delta u - u = f(x,u)$ is the Euler-Lagrange equation for a critical point of J. Let

$$\sigma = \inf J(\varphi), \qquad \varphi \in H^{2,1}(M).$$

The main step is to prove that J has a minimum, and hence a critical point.

Theorem 5.2 *There is a function $u \in H^{2,1}(M)$ minimizing J. Moreover, if $f(x,s) \in C^\infty$ then $u \in C^\infty$, and u is a solution of $\Delta u - u = f(x,u)$. In view of Lemma 5.1, if $g \in C^\infty$ satisfies condition* (5.7)*, then there is a solution $u \in C^\infty$ of $\Delta u = g(x,u)$.*

Proof. Step 1 is to show that J is bounded below. Because f is bounded, $|f(x,s)| \le A$ for some constant A. Thus by (5.11), for any constant $\epsilon > 0$

$$2|F(x,s)| \le 2As \le \epsilon s^2 + \frac{1}{\epsilon}A^2.$$

Choosing $\epsilon = 1/2$ we conclude from (5.11) that

$$J(u) \ge \int_M (|\nabla u|^2 + \tfrac{1}{2}u^2)\,dx_g - 2A^2\,\mathrm{Vol}(M) \tag{5.13}$$

that implies J is bounded below, so $\sigma > -\infty$ and there is a sequence of functions $u_j \in H^{2,1}(M)$ with $J(u_j) \downarrow \sigma$.

Step 2 is to show that, in some sense, the sequence u_j has a convergent subsequence. From (5.13) and the fact that $J(u_j) \le J(u_1)$, it is clear that

$$\|u_j\|^2_{H^{2,1}} \le 2\int_M (|\nabla u_j|^2 + \tfrac{1}{2}u_j^2)\,dx_g \le \text{constant}.$$

As we used in our discussion of the eigenvalues of the Laplacian, Chapter 3.3, a closed ball in a Hilbert space, such as $H^{2,1}$, is weakly compact, so a subsequence of the u_j (which we relabel u_j) converges weakly to some $u \in H^{2,1}$.

Although the functional $J(u_j)$ is not continuous under weak convergence, it is *lower semicontinuous*; this is adequate. In greater detail, first, since norms are lower semicontinuous under weak convergence (see 3.9), we know that $\|u\| \le \liminf \|u_j\|$. Further, the Sobolev Embedding Theorem 1.1 tells us that if $p < 2n/(n-2)$ the embedding $H^{2,1} \hookrightarrow L^p$ is compact. Because compact linear maps take weakly convergent sequences into (strongly) convergent ones, we see that $u_j \to u$ strongly in L^p for any $p < 2n/(n-2)$, especially in L^1 and L^2. Thus

$$\int_M u_j^2\,dx_g \to \int_M u^2\,dx_g$$

and, by the mean value theorem

$$\int_M |F(x,u_j) - F(x,u)|\,dx_g \le A\int_M |u_j - u|\,dx_g \to 0.$$

Consequently

$$J(u) = \|u\|^2_{H^{2,1}} + \int_M 2F(x,u)\,dx_g \le \liminf J(u_j) = \sigma.$$

But by definition of σ, $J(u) \ge \sigma$. Thus $J(u) = \sigma$ so $u \in H^{2,1}$ is the desired function minimizing J.

Step 3 consists of showing that this function $u \in H^{2,1}$ is actually smooth if f is smooth. Since u minimizes $J(u)$, then for any $z \in H^{2,1}$ the function $\Psi(\epsilon) = J(u + \epsilon z)$ has a minimum at $\epsilon = 0$. Thus $\Psi'(0) = 0$, that is, for any $z \in H^{2,1}(M)$

$$\int_M [\nabla u \cdot \nabla z + uz + f(x,u)z] \, dx_g = 0, \tag{5.14}$$

so $u \in H^{2,1}(M)$ is a weak solution (see Chapter 4.2) of $\Delta u - u = f(x,u)$. At this point, we can refer to general results ([GT, Theorem 8.8] and Theorem 2.3 above) to conclude that $u \in C^\infty$ —and thus satisfies the equation $\Delta u - u = f$ as one can see after integrating the first term in (5.14) by parts.

There is an alternate procedure to prove that u is smooth, one we also used earlier (Section 3.3). Let $k(x) = f(x,u)$. This is in L^∞ since $f(x,s)$ is bounded. Thus there is a unique solution $v \in H^{p,2}$ of the linear equation $\Delta v - v = k$ for all $p > 1$. For $p \geq 2$ clearly $v \in H^{2,2} \subset H^{2,1}$ and satisfies the linear equation

$$\int_M [\nabla v \cdot \nabla z + vz + kz] \, dx_g = 0 \tag{5.15}$$

for any $z \in H^{2,1}$. Note that (5.14) states that u is also a solution of (5.15), so $w = u - v \in H^{2,1}$ satisfies

$$\int_M (\nabla w \cdot \nabla z + wz) \, dx_g = 0 \tag{5.16}$$

for any $z \in H^{2,1}$. By choosing $z = w$ we see that $|\nabla w|^2 + |w|^2 = 0$ and hence $w = 0$, that is, $u = v$. But $v \in H^{p,2}$ for all p so $u \in H^{p,2}$ for all p. Consequently $k(x) = f(x,u) \in C^{1,\alpha}$ (pick $p > n$) and hence $u = v \in C^{3,\alpha}$. Continuing by induction, $u \in C^\infty$. □

As another type of application, one that we will need later on, we investigate the lowest eigenvalue λ_1, and corresponding eigenfunction φ_1 of

$$Lu = -\Delta u + c(x)u, \tag{5.17}$$

where $c(x) \in C^\infty$ is a given function. The lowest eigenvalue λ_1, with corresponding eigenfunction φ, satisfies $L\varphi = \lambda_1 \varphi$. We already treated the special case $c(x) = 0$ in Section 3.3. As before, multiplying this by φ and integrating by parts, we find that λ_1 is given by the *Rayleigh quotient*

$$\lambda_1 = \min \frac{\int_M (|\nabla \varphi|^2 + c(x)\varphi^2) \, dx_g}{\int_M \varphi^2 \, dx_g}; \tag{5.18}$$

Multiplying φ by a constant, we can assume that $\|\varphi\|_{L^2} = 1$ and hence the lowest eigenvalue[1], value of the functional

$$J(u) = \int (|\nabla u|^2 + cu^2) \, dx_g \qquad \text{on} \qquad \|u\|_{L^2} = 1.$$

The approach used in Section 3.3 for the higher eigenvalues proves the existence of an eigenfunction φ, minimizing J on $\|\varphi\|_{L^2} = 1$. In this case, the eigenfunction φ_1

[1]Caution: if $c = 0$, then clearly $\varphi =$ constant minimizes J and $\lambda_1 = 0$. In this special case, as we did in Section 3.3, one usually relabels the λ_j and writes $\lambda_0 = 0$ and then calls $\lambda_1 > 0$ the "lowest non-trivial eigenvalue". Mathematicians are inconsistent in this numbering. It can be confusing.

corresponding to the lowest eigenvalue is not a constant (unless $c(x) \equiv$ const. However, we will show that φ_1 is never zero, so (multiplying by -1 if necessary) we have $\varphi_1 > 0$; this is analogous to the positivity of the eigenfunction corresponding to the lowest note of a drum. Since $\varphi \in H^{2,1}$ then $\psi \equiv |\varphi| \in H^{2,1}$ and $|\nabla\varphi| = |\nabla\psi|$ almost everywhere ([Au-4], page 82). Thus $\psi \in A$ and $J(\psi) = J(\varphi)$ so ψ also minimizes J on A. Therefore by the above reasoning $\psi \in C^\infty(M)$ and is also an eigenfunction of L with eigenvalue λ_1

$$-\Delta\psi + c\psi = \lambda_1\psi.$$

Pick a constant $\gamma > 0$ so that $\lambda_1 - c + \gamma > 0$. Then because $\psi \geq 0$ we find that

$$-\Delta\psi + \gamma\psi = (\lambda_1 - c + \gamma)\psi \geq 0. \tag{5.19}$$

The strong maximum principle (see section 2.6) states that under these conditions either $\psi \equiv 0$ or else $\psi > 0$ everywhere. Since $\int_M \psi^2\,dx_g = 1$, the only possibility is that $\psi > 0$. Because $\psi = |\varphi|$ this also implies that either $\varphi > 0$ or $\varphi < 0$ everywhere, so any eigenfunction with eigenvalue λ_1 is either positive or negative. The eigenspace is then one dimensional, for if the dimension were two or more, then there would be two orthogonal eigenfunctions φ, ψ with eigenvalue λ_1. However the orthogonality condition $\int_M \varphi\psi\,dx_g = 0$ is impossible because $\varphi\psi$ never changes sign. The next proposition collects these facts.

Proposition 5.3 *Let $Lu = -\Delta u + cu$. Then the eigenspace corresponding to the lowest eigenvalue, λ_1, is one dimensional and the corresponding eigenfunctions are never zero; in particular, there is a positive eigenfunction $\varphi_1 > 0$ of $L\varphi_1 = \lambda_1\varphi_1$.*

One can also give a very different proof of this result using the Krein-Rutman (see [KR] and [Kr]) generalization of the Perron-Frobenius theory of positive matrices. In this case, the maximum principle gives the positive operator. An advantage of this alternate proof is that it applies to second order elliptic operators that are *not* necessarily self-adjoint.

Before closing our discussion on the calculus of variations, we should mention that there are techniques such as the "Mountain Pass Lemma" and generalizations of finite dimensional Morse Theory for proving the existence of saddle points (i.e. critical points that are not local minima).

5.3 Continuity Method

The idea here is quite simple. Say one wishes to solve some equation $F(u) = 0$. Consider a family of problems depending continuously on a parameter t

$$P_t: \qquad F(u,t) = 0, \qquad 0 \leq t \leq 1,$$

where $F(u,1) = F(u)$ is the desired problem and $F(u,0) = 0$ is some equation that you know how to solve. Let

$$A = \{t \in [0,\, 1] : \text{ one can solve } P_t\}.$$

By choice of P_0 we know $0 \in A$ so A is not empty.

One shows that A is both open and closed. The proof that A is open customarily uses the implicit function theorem: if $F(u_0, t_0) = 0$, then solve $F(u,t) = 0$ for all t near t_0. Of course, one must verify the assumptions of the implicit function theorem.

To prove that A is closed, say $t_j \in A$ and $t_j \to \tau$; we must show that $\tau \in A$. Now $t_j \in A$ means there are solutions u_j of $f(u_j, t_j) = 0$. The goal is to find a convergent subsequence of the u_j, say $u_j \to u$, since then, by continuity, $F(u, \tau) = \lim F(u_j, t_j) = 0$ so $\tau \in A$ as desired.

If $F(u, t) = 0$ is a second order partial differential equation for u, then uniform convergence of (a subsequence of) the $u_j \to u$ in $C^2(M)$ is enough. To obtain this, one often uses the Arzela-Ascoli lemma; consequently we would like to prove that any solution u of problem P_t satisfies $\|u\|_{C^{2,\alpha}} \le$ *const.*, where the constant is independent of $t \in [0, 1]$. Obtaining this basic *a priori* estimate on solutions of P_t is usually the most difficult step in the continuity method. If one uses spaces other than $C^{2,\alpha}$, then one replaces the Arzela-Ascoli lemma by an appropriate compactness lemma.

The first step is to define the problems P_t. There are usually many ways. For (5.1) with $a > 0$ and $h > 0$ we consider the family of problems

$$F(u, t) := \Delta u - hu + t[a - h(e^u - u)] = 0, \qquad 0 \le t \le 1. \tag{5.20}$$

(We could also use $\Delta u + a - [th + (1-t)a]e^u = 0$, which is more natural for geometric reasons, but we use (5.20) to save work in Section 5.4). At $t = 0$, a solution is $u = 0$. Let A be the set

$$A = \{t \in [0, 1] : \ F(u, t) = 0 \text{ has a solution } u \in C^2(M)).$$

To prove A is open we use the implicit function theorem. Say $F(u_0, t_0) = 0$. Then the linearization of (5.20) at u_0, t_0 is

$$Lv = F_u(u_0, t_0)v = \Delta v - h(1 - t_0 + t_0 e^{u_0})v.$$

Because $h > 0$ and $(1 - t + te^u) > 0$ for $0 \le t \le 1$, by part *b)* of the Comparison Theorem 4.4 $\ker L = \ker L^* = 0$. Thus the Fredholm alternative tells us that L is an isomorphism from $H^{p,2}$ to L^p and also from $C^{2,\alpha}$ to C^α. Since we need the map $F(u, t)$ to be a C^1 map, if we use the space $H^{p,2}$ we require that $p > n/2$ because then $H^{p,2}(M) \hookrightarrow C^0(M)$. In any case, standard elliptic regularity shows that $u \in C^\infty$ so openness is proved.

Next we prove that A is closed by proving an *a priori* inequality. Pick any $0 < \alpha < 1$; we want a constant R independent of $t \in [0, 1]$ so that any solution $u \in C^{2,\alpha}(M)$ of (5.20) is in the ball

$$\|u\|_{C^{2,\alpha}} \le R. \tag{5.21}$$

To prove this, we first prove an estimate in C^0 using the maximum principle. At a maximum of u, from (5.20)

$$0 \ge \Delta u = h[te^u + (1-t)u] - ta.$$

But $e^s \ge 1 + s$ for all real s (e^s lies above its tangent line at $s = 0$). Thus

$$u \le t + u \le te^u + (1-t)u \le \max_M [a(x)/h(x)].$$

Using a similar estimate for the minimum of u we have an *a priori* uniform estimate for any solution of (5.20)

$$\|u\|_{C^0} \le \text{constant independent of } t \in [0, 1]. \tag{5.22}$$

To estimate the higher order derivatives of u we use the basic inequalities (2.9) for linear elliptic operators to conclude from (5.20) to (5.22) that

$$\|u\|_{H^{p,2}} \le c_1\|\Delta u\|_{L^p} + c_2\|u\|_{L^1} \le c_3. \tag{5.23}$$

Pick $p > n$. Then Sobolev Inequality (1.26) shows that $\|u\|_{C^1(M)} \le c_4$, and hence, by (5.20) and the Schauder Estimates (2.8) we have

$$\|u\|_{C^{2,\alpha}} \le c_5\|\Delta u\|_{C^\alpha} + c_6\|u\|_{C^0} \le c_7.$$

Armed with this estimate, one can apply the Arzela-Ascoli lemma and conclude that the set A is closed.

You are invited to use the continuity method to solve (5.2).

5.4 Schauder Fixed Point Theorem

EXAMPLE 5.1 Is there a solution (x, y) of the system of the "high school" equations

$$\left.\begin{aligned} 3x + 2y &= \frac{x^2 + e^{\sin xy}}{1 + 3x^2 + y^{16}} \\ 4x - 5y &= \frac{7 + \sin(x + y^3)}{1 + e^{x-y}} \end{aligned}\right\} \qquad ?$$

This is a special case of $AX = F(X)$, where A is an invertible matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ a bounded continuous map, $\|F(x)\| \le c_1$. If one rewrites this as $X = A^{-1}F(X)$, then the Brouwer fixed point theorem can be used as follows to prove a solution exists. Any solution X of this must satisfy the *a priori* inequality

$$\|X\| = \|A^{-1}F(X)\| \le \|A^{-1}\|c_1 = c_2,$$

Pick some $R > c_2$ and let $B_R = \{\|X\| \le R\}$. Then by the Brouwer theorem the map $G = A^{-1} \circ F$ maps the ball B_R to itself and hence has a fixed point. As an exercise you may find it amusing to get the same conclusion assuming that F grows slower than linearly, that is, $\lim_{\|X\|\to\infty}\|F(X)\|/\|X\| \to 0$ instead of assuming F is bounded. □

REMARK 5.2 This result would be difficult to prove by the continuity method as described in the preceding section. The problem is in using the implicit function theorem to prove the "openness", since we have made no assumptions about the derivative of $F(X)$. One might, however, prove the openness by some other procedure—which is essentially what we did by using the fixed point theorem.

As a digression, we will give a short proof of the Brouwer fixed point theorem using Stokes' theorem. Our key step is the "no-retract" theorem. Let M be an n-dimensional smooth connected orientable compact manifold with smooth boundary, ∂M. A map $f : M \to \partial M$, is called a *retraction* if f is the identity map on the boundary, ∂M. We claim that there can not be a smooth retraction (there cannot even be a continuous retraction).

First some background. Let N be a smooth k-dimensional compact manifold N without boundary. with a volume form ω, so ω is a k-form. For orientable N one can obtain ω in many ways, such as by introducing a Riemannian metric on N. For any k-form on N, we have $d\omega = 0$; if $f : M \to N$ is a smooth map, then $d(f^*\omega) = f^*(d\omega) = 0$.

Apply this to the special case when N is ∂M. Then by Stokes' theorem we have

$$\int_{\partial M} f^*\omega = \int_M d(f^*\omega) = \int_M f^*(d\omega) = 0.$$

If f is a retraction it is the identity on ∂M so $\omega = f^*\omega$ there. Consequently the integral above on the left is $\int_{\partial M} \omega = \mathrm{Vol}(\partial M)$, which cannot be zero. This contradiction proves that the retraction f cannot exist. [To check your understanding, note that there is no contradiction if f is just a diffeomorphism of M that leaves the boundary fixed. There is also no contradiction if f maps all of M to one point on its boundary. It is important that f map *all* of M to its boundary, keeping the boundary fixed pointwise.]

Using the no-retract theorem we follow a standard proof of the Brouwer theorem that any continuous map f from a closed ball B in $\mathbb{R}^n$ to itself must have at least one fixed point. First assume f is smooth. If it has no fixed point then for each $x \in B$ the vector $V(x) = f(x) - x$ from x to $f(x)$ is never zero. Consider the straight line $\gamma(t) = x + tV(x)$ passing through x and $f(x)$. Let p be the point on the boundary *backward* beyond x (so $t \le 0$) where this line meets the boundary (draw a figure). Define the map $\varphi : B \to \partial B$ by the rule $\varphi : x \mapsto p$. Then φ is the identity map on the boundary and hence is a retraction from B to ∂B. But we just proved that such a map cannot exist. Thus a smooth f must have a fixed point. If f is only continuous, then approximate it by smooth maps $f_j(x)$ whose fixed points x_j (or a subsequence) converge to a fixed point of f. □

The Schauder fixed point theorem allows us to apply the procedure of the Example 5.1 and solve some nonlinear elliptic equations. Before attending to that, we present an example of a continuous map f from the unit ball in Hilbert space into its boundary. This map will have *no* fixed point. It shows that any generalization of the Brouwer fixed point theorem to infinite dimensional spaces will need some extra assumption, either on the map or on the set S to which the map is applied.

EXAMPLE 5.2 Let H be the Hilbert space ℓ_2 of sequences $x = (x_1, x_2, \ldots)$ with $|x|^2 = \sum |x_j|^2 < \infty$ and let S denote the closed unit ball $\{|x| \le 1\}$. The *continuous map* $f : x \mapsto (\sqrt{1 - |x|^2}, x_1, x_2, \ldots)$ does map S into S, but *does not have a fixed point* (since $|f(x)| = 1$ for all $x \in S$, at a fixed point $|x| = 1$ which implies the incompatible assertions $x_1 = \sqrt{1 - |x|^2} = 0$, $x_2 = x_1 = 0$, $x_3 = x_2 = 0$, etc.). □

The Schauder theorem makes a compactness assumption that avoids the difficulties of this example. Let B be a Banach space and $S \subset B$. A continuous map $f : S \to B$ is called *compact* if the images of bounded subsets of S are precompact (that is, for any bounded set $Q \subset S$, the closed set $\overline{f(Q)}$ is compact).

Theorem 5.4 [SCHAUDER FIXED POINT THEOREM] *Let B be a Banach space and $S \subset B$ a convex, closed, bounded subset. If $f : S \to S$ is a compact map, then f has a fixed point.*

See [GT] or [N-3] for a short proof. The idea is to find finite dimensional approximations to which the Brouwer theorem applies. This gives approximate fixed points, x_k. The compactness of f enables one to find a convergent subsequence to an honest fixed point of f.

One very useful corollary is the following result. It was first found by Leray-Schauder using their extension to Banach spaces of the Brouwer degree of a map (see [N-3] a discussion of the degree). There is now a short direct proof using only the Schauder Fixed Point Theorem (see [GT]).

Theorem 5.5 [LERAY-SCHAUDER]. *Let B be a Banach space and $F: B\times[0,1]\to B$ a compact mapping with $F(x,0)=0$ for all $x\in B$. Assume there is a constant c such that any solution $(x,t)\in B\times[0,1]$ of $x=F(x,t)$ satisfies the a prior inequality $\|x\|\le c$. Then the map $F(x)=F(x,1): B\to B$ has a fixed point.*

This theorem shows clearly that if one has a good *a priori* estimate on the solutions of an equation, then one can prove the existence of a solution.

As our first application of these fixed point theorems we use the Schauder fixed point theorem 5.4 to prove the existence of a solution of

$$Lu := \Delta u - u = f(x,u,\nabla u), \tag{5.24}$$

where $f(x,u,p)$ is a bounded smooth function of all its variables. In view of Lemma reflemma:TECH.7 this will prove there is a solution of $\Delta u = g(x,u)$ assuming g satisfies the condition (5.7), and hence a solution of (5.1) and (5.2). We simply copy our discussion of the model equation $AX=F(X)$ in R^n and observe that by the Fredholm alternative 2.4, the linear map $Lu := \Delta u - u$ is an isomorphism from $H^{p,2}\to L^p$ and also from $C^{2,\alpha}\to C^{\alpha}$. Thus, we solve

$$u = L^{-1}f(x,u,\nabla u).$$

It is natural to let $T(u)=L^{-1}f(x,u,\nabla u)$. Now $f: C^{1,\alpha}\to C^{\alpha}$ and $L^{-1}: C^{\alpha}\to C^{2,\alpha}$. Moreover, the by the Arzela-Ascoli Theorem, the identity map $id: C^{2,\alpha}\hookrightarrow C^{1,\alpha}$ is a compact operator. Thus the map $T: C^{1,\alpha}\to C^{1,\alpha}$ defined by the composition

$$C^{1,\alpha}\xrightarrow{f} C^{\alpha}\xrightarrow{L^{-1}} C^{2,\alpha}\overset{id}{\hookrightarrow} C^{1,\alpha} \tag{5.25}$$

is compact. In addition, since $f(x,u,\nabla u)$ is a bounded function, using the basic estimate (2.8) there is a constant K such that

$$\|T(u)\|_{C^{1,\alpha}(M)}\le K$$

for all $u\in C^{1,\alpha}(M)$. Thus let

$$S=\{u\in C^{1,\alpha}(M):\ \|u\|_{C^{1,\alpha}(M)}\le K\}.$$

The Schauder theorem 5.4 and elliptic regularity prove the following.

Theorem 5.6 *Let $f(x,s,p): M\times\mathbb{R}\times TM\to\mathbb{R}$ be a bounded smooth function. Then the equation $\Delta u - u = f(x,u,\nabla u)$ has at least one smooth solution.*

In (5.25) we could have also used Sobolev spaces $T: H^{p,1}\to H^{p,1}$ for any $p>n$. (We need $p>n$ to insure that T is continuous.) As an exercise, one can also prove this result by applying the Leray-Schauder theorem 5.5 to the family of equations

$$\Delta u - u = tf(x,u,\nabla u), \quad \text{where} \quad 0\le t\le 1.$$

For variety, we will now use the Leray-Schauder Theorem 5.5 to solve (5.1) directly. We consider the family of equations for $0\le t\le 1$

$$Lu := \Delta u - hu = t[-a+h(e^u-u)] = tf(x,u) \tag{5.26}$$

(at $t = 1$ this is (5.1)). Because L is invertible between the usual spaces, we write this equation as

$$u = tL^{-1}f(x, u).$$

Comparing this equation with Corollary 5.5 it is reasonable to let $F(u, t) = tL^{-1}f(x, u)$. By the reasoning we used for (5.24), it is evident that $F : C^{1,\alpha} \to C^{1,\alpha}$ is a compact map. Thus, we need only establish the *a priori* inequality

$$\|u\|_{C^{1,\alpha}(M)} \leq \text{ constant} \tag{5.27}$$

for any solution u of $u = tL^{-1}f(x, u)$, that is, any solution u of (5.24), $0 \leq t \leq 1$. But equation (5.26) is exactly the equation (5.20) we (deliberately) used for the continuity method and (5.27) is a consequence of (5.21) so the proof is completed.

5.5 Sub and Supersolutions

The simplest version of this method goes as follows. We say that u_+ is a *supersolution* and u_- a *subsolution* of $\Delta u = f(x, u)$ if, respectively,

$$\Delta u_+ \leq f(x, u_+), \qquad \text{and} \qquad \Delta u_- \geq f(x, u_-). \tag{5.28}$$

For the Laplace equation, $\Delta u = 0$, subsolutions are simply subharmonic functions.

Theorem 5.7 *Let $f(x, s) \in C(M \times \mathbb{R})$. If there are sub and supersolutions $u_\pm \in H^{p,2}(M)$, $p > n$, and if $u_-(x) \leq u_+(x)$, then there is at least one solution $u \in H^{p,2}(M)$ of $\Delta u = f(x, u)$ in the interval $u_-(x) \leq u(x) \leq u_+(x)$.*

The proof is a simple iteration procedure using the maximum principle (see [KW-1] for a short exposition). A generalization using the Schauder Fixed Point Theorem is in [CBL]. These proofs use the fact that one can solve certain linear elliptic equations. For equations with severe nonlinearities, one can often prove a version of Theorem 5.7 (see [Au-4, Chapter 7, Section 12] and [CNS]) but one must already know some non-trivial existence result. (Theorem 5.7 is also true for complete, non-compact manifolds, as well as for boundary value problems—although for boundary value problems one must slightly modify it.)

EXAMPLE 5.3 This method gives the shortest existence proofs for (5.1) and (5.2). Indeed, for a subsolution in (5.1) try $u_-(x) = \alpha$, where α is a constant. Then from (5.28) we need

$$0 \geq -a + he^{\alpha},$$

which will clearly be satisfied by choosing α to be a sufficiently large negative constant. Similarly, any sufficiently large positive constant $u_+(x) = \beta$ will be a supersolution of (5.1). Thus, there is a solution $u_- \leq u \leq u_+$. The same easy proof works for (5.2) with $u_- =$ small constant > 0; then, since the solution satisfies $u(x) \geq u_-(x)$ we are assured that $u > 0$. □

EXAMPLE 5.4 There may be many sub- and supersolution pairs, $u_- \leq u_+$. For instance, consider

$$\Delta u = f(x, u) + \cos u,$$

where $|f(x, s)| \leq 1$ for all x and s. The functions $u(x) = 2k\pi$ are all supersolutions, while $u(x) = (2k - 1)\pi$ are subsolutions. Hence there is at least one solution in each of the intervals $(2k - 1)\pi \leq u(x) \leq 2k\pi$. □

Here are two more general applications of the method. For simplicity, assume $f(x,s) \in C^\infty(M \times \mathbb{R})$. The first result extends the linear existence theory (2.14) for $\Delta u = f(x)$.

Theorem 5.8 [KW-4] *Assume* $\partial f(x,s)/\partial s \geq 0$. *Then there exists a solution* $u \in C^2(M)$ *of*

$$\Delta u = f(x,u) \tag{5.29}$$

if and only if there is a function $v \in C^2(M)$ *satisfying*

$$\int_M f(x, v(x))\, dx_g = 0. \tag{5.30}$$

Proof. For the necessity, integrate (5.29) to see that any solution of (5.29) satisfies (5.30). To prove the sufficiency, given $v(x)$ let $\varphi(x) = f(x, v(x))$. Because $\int_M \varphi\, dx_g = 0$ there is a solution z of $\Delta z = \varphi(x)$. Let $u_+ = z + c_+$, where the constant c_+ is chosen so that $u_+ \geq v$. Then

$$\Delta u_+ = f(x, v(x)) \leq f(x, u_+(x)).$$

Similarly, let $u_- = z + c_-$. □

As an example where the assumptions are satisfied, we consider equation (5.1) with $h \geq 0 (\not\equiv 0)$ and conclude that there is a solution if and only if the coefficient $a(x)$ satisfies $\int_M a(x)\, dx_g > 0$.

The second application mildly generalize some results proved above. Consider the equation

$$\Delta u = f(x,u) + g(x,u), \tag{5.31}$$

where f and g are smooth functions with $g(x,s)$ bounded and f having the property that

$$\frac{\partial f}{\partial s}(x,s) \geq \gamma(x) \qquad \text{for all real } s \tag{5.32}$$

for some smooth function $\gamma(x) \geq 0 (\not\equiv 0)$.

Theorem 5.9 *Assume that* $f(x,s)$ *satisfies* (5.32) *and* $g(x,s)$ *is a bounded function. Then there exists a solution of* (5.31).

Proof. Let λ_1 be the lowest eigenvalue of $L\varphi := -\Delta\varphi + \gamma\varphi$. Then $\gamma \geq 0\, (\not\equiv 0)$ so the Rayleigh quotient (5.18) shows that $\ker L = 0$ and $\lambda_1 > 0$. Moreover, by Proposition 5.3 there is a positive eigenfunction φ of $L\varphi = \lambda_1 \varphi$. Say $|g(x,s)| \leq A$ and let z be the unique solution of the linear equation

$$\Delta z - \gamma z = f(x,0) - A$$

Choose the constant c_+ so large that $u_+ = z + c_+\varphi > 0$. Then u_+ is a supersolution of (5.30). Similarly, if v is the solution of $\Delta v - \gamma v = f(x,0) + A$, then for sufficiently large negative c_-, the function $u_- = v + c_-\varphi < 0$ and is a subsolution. □

The above idea of using the (positive) lowest eigenfunction of a linear problem to construct sub or supersolutions of a nonlinear problem is a useful device. In particular for equation (5.2) it is often useful to consider the lowest eigenvalue λ_1 and corresponding eigenfunction $\varphi > 0$ (by Proposition 5.3 for the operator $Lu = -\Delta u - au$, so $L\varphi = \lambda_1\varphi$. Using both sub and supersolutions of the form $u_\pm = c_\pm \varphi$, where $0 < c_- < c_+$, one can quickly prove that if $h > 0$ then (5.2) has a positive solution if and only if $\lambda_1 < 0$. This is a weaker assumption than our earlier one that $a > 0$ since the Rayleigh quotient (5.18) with $\psi = 1$ shows that $a > 0$, and even the weaker condition $\int_M a(x)\, dx_g > 0$, implies that $\lambda_1 < 0$.

5.6 The Heat Equation

Another technique that is useful for solving equations such as $\Delta u = f(x,u)$ is to solve the *initial value problem* for the *heat equation*

$$\frac{\partial u}{\partial t} = \Delta u - f(x,u) \quad \text{for} \quad t > 0, \quad x \in M, \tag{5.33}$$

$$u(x,0) = u_0(x), \qquad x \in M, \tag{5.34}$$

where u_0 is some prescribed function, and show that as $t \to \infty$ then the $u(x,t)$ converge to some function $v(x)$ which satisfies $\Delta v = f(x,v)$, that is, $v(x)$ is an "equilibrium solution" of the heat equation (5.33). (Actually, it is enough to show that some subsequence, $u(x,t_j)$ converges as $t_j \to \infty$.)

There are three steps when using this method.

Step 1. Show that a solution of (5.33)–(5.34) exists for short times $0 \le t \le \epsilon$. In this regard, one should note that the *backward heat equation*, $-u_t = \Delta u$, $u(x,0) = u_0(x)$ does *not* necessarily have a solution for short time.

Step 2. Show that a solution of (5.33)-(5.34) exists for *all* time, $0 \le t < \infty$. The simple ordinary differential equation $u_t = u^2$ with initial condition $u(0) = c$, has the unique solution $u(t) = c/(1-tc)$; this shows that a solution of an innocent-looking equation may not exist for all time. Moreover, it shows that—for nonlinear equations—the maximal interval for which a solutions exists may depend upon the initial conditions.

Step 3. Prove that as $t \to \infty$, then $u(x,t)$ (or $u(x,t_j)$) converges to a solution $v(x)$ of $\Delta v = f(x,v)$. The following example shows that some hypotheses will be needed. One just observes that $u(x,t) = e^{3t}\cos x$ satisfies the heat equation $u_t = u_{xx} + u$ on S^1 but u has no limit as $t \to \infty$. Another example is the bounded function $w(x,t) = \cos(x+t)$ which, also on the circle S^1, satisfies the heat equation $w_t = w_{xx} + w_x + w$. Neither $u(x,t)$ nor $w(x,t)$ converge to anything for large t.

We will carry out these steps for

$$\frac{\partial u}{\partial t} = \Delta u + a(x) - h(x)e^u \tag{5.35}$$

with the initial condition

$$u(x,0) = \varphi(x). \tag{5.36}$$

As usual, the same ideas prove a more general result whose formulation and proof we leave as an exercise; one version is suggested at the end of this section.

Theorem 5.10 *Assume that* $a > 0$, $h > 0$, *and* φ *are any smooth functions. Then there exists a unique solution* $u(x,t)$ *of* (5.35)–(5.36) *for all* $t > 0$. *Moreover, there is a function* $v \in C^\infty(M)$ *so that*

$$\lim_{t\to\infty} u(x,t) = v(x), \tag{5.37}$$

and v *satisfies the "equilibrium" equation* $\Delta v + a - he^v = 0$.

Proof. Step 1. The existence of a unique solution for a small time interval, $0 \le t < \epsilon$, is a consequence of Theorem 4.6.

Step 2. To prove that the solution exists for all time $0 \le t < \infty$, we need to estimate the solution. Let $0 \le t < T$ be a *maximal interval* on which a solution exists and pick any $0 < T_0 < T$. Consider the maximum value of $u(x,t)$ on the compact set $M \times [0, T_0]$ and say the maximum is at some point $p = (x_0, t_0)$. If $t_0 = 0$, then $u(x,t) \le u(x,0) \le$ *const.*, while if $t_0 > 0$ then $u_t(p) \le 0$ (in fact, $u_t(p) = 0$ if $0 < t_0 < T_0$), $\nabla u(p) = 0$, and $\Delta u(p) \le 0$. Thus from (5.35), $h(p)e^{u(p)} \le a(p)$ so $u \le \max_M [\log(a(x)/h(x))]$. Looking at the point where u has its minimum we get a similar lower bound for u. Since these estimates are independent of T_0,

$$|u(x,t)| \le m, \tag{5.38}$$

where the constant m does not depend on T.

Next we would like to estimate the derivatives of u. One approach uses the Sobolev space analog of (2.33) combined with the Sobolev inequality (just as we used (5.23) to estimate the $\|u\|_{C^1}$ in Section 5.3). Another way is to proceed directly, rewriting (5.35) as $u_t = \Delta u + f(x,t)$ where, using (5.38), $f = a - he^u$ is now known to be a bounded continuous function, so one can apply simpler local estimates for the heat equation itself to conclude that

$$\|u\|_{C^1(M\times[0,\,T])} \le m_1,$$

with m_1 independent of T. Repeatedly applying the estimate (2.33) we find that for all $x \in M$ and $0 \le t < T$

$$|\partial_t^r \partial_x^s u(x,t)| \le m_{r,s}, \tag{5.39}$$

where the constants $m_{r,s}$ are independent of T.

If $T < \infty$, we will show that as $t \to T$, then $u(x,t)$ and all of its derivatives converge uniformly to some function, which we call $u(x,T)$. But then we can apply Theorem 4.6 to solve (5.30) on some interval $T \le t \le T + \epsilon$ using $u(x,T)$ as the initial value. This will define a smooth solution $u(x,t)$, for the larger interval $0 \le t \le T + \epsilon$, of (5.30) and contradicts the maximality of T.

To prove the convergence of $u(x,t)$, we use the mean value theorem (in a local coordinate chart) to estimate

$$\begin{aligned} |u(x,t) - u(y,t')| &\le |u(x,t) - u(x,t')| + |u(x,t') - u(y,t')| \\ &\le \|\partial_t u\|_\infty |t - t'| + \|\partial_x u\|_\infty |x - y|. \end{aligned}$$

Therefore, in view of the estimates (5.39), the function $u(x,t)$ is uniformly continuous on $M \times [0,T)$ and hence has a unique continuous extension to $M \times [0,\,T]$. Replacing u by $\partial_t^r \partial_x^s u$ in the above, we find that we can extend u and all of its derivatives continuously to $M \times [0,\,T]$.

Step 3. Now we know the solution $u(x,t)$ exists for all $t \ge 0$ and must show it converges as $t \to \infty$. First we show that

$$|u_t(x,t)| \le ke^{-\gamma t} \tag{5.40}$$

for some positive constants k and γ. To see this, differentiate (5.35) to find that $w = u_t$ satisfies

$$w_t = \Delta w - bw,$$

where $b = he^u > \gamma > 0$ for some constant γ (again we used (5.38)). Then the function $z(x,t) = w(x,t)e^{\gamma t}$ satisfies

$$z_t = \Delta z - cz,$$

with $c = b - \gamma > 0$. But for this equation, a direct argument as in Step *2* above (looking at a positive minimum of z), or else the maximum principle, Theorem 2.13, shows that

$$\begin{aligned} |z(x,t)| &\le \max_M |z(x,0)| \;=\; \max_M |w(x,0)| \\ &= \max_M |\Delta u + a - he^u|\Big|_{t=0} \le\; k \end{aligned}$$

For some constant k. Since $u_t = ze^{-\gamma t}$, this gives the estimate (5.40).

Let T be a constant to be determined and say $T \le t' \le t$. Then from (5.40) we have

$$|u(x,t) - u(x,t')| = |\int_{t'}^{t} u_t(x,s)\,ds| \le \frac{k}{\gamma} e^{-\gamma T}. \tag{5.41}$$

Write $\varphi(x,t) = -a + he^u + u_t$, so the equation (5.30) is

$$\Delta u = \varphi. \tag{5.42}$$

From the estimate (5.40) and (5.41), we see that $\varphi(x,t)$, viewed as a sequence of functions with t as a parameter, is uniformly Cauchy as $t \to \infty$. Hence by the $H^{p,2}$ estimates (2.9) for the elliptic operator Δ, given any $\epsilon > 0$ we have

$$\begin{aligned} \|u(\cdot,t) - u(\cdot,t')\|_{H^{p,2}(M)} \quad \le \quad & \text{const.}\|\varphi(\cdot,t) - \varphi(\cdot,t')\|_{L^p(M)} \\ & +\text{const.}\|u(\cdot,t) - u(\cdot,t')\|_{L^p(M)}. \end{aligned} \tag{5.43}$$

This can be made as small as we wish by picking $T \le t' \le t$ with T sufficiently large. Choosing $p > n$ and combining (5.43) with the Sobolev inequality we conclude that $u(x,t)$ is Cauchy in $C^1(M)$ as $t \to \infty$. The estimate (5.43), only now using the Hölder norms (2.8), shows that u is Cauchy in $C^{2,\alpha}(M)$ as $t \to \infty$. Repeatedly using (2.8) we find that u is Cauchy in $C^k(M)$, for all k, to some function $v(x) \in C^\infty(M)$. Passing to the limit $t \to \infty$ in (5.30) we complete the proof. □

As an exercise, one may find it useful to generalize this proof to $u_t = \Delta u - f(x,u)$, assuming that $f_s(x,s) > 0$, $f(x,\infty) > 0$, and $f(x,-\infty) < 0$.

5.7 Summary for $\Delta u = f(x) - k(x)e^u$

Since we have spent this whole chapter discussing $\Delta u = -a + he^u$, assuming $a > 0$ and $h > 0$, we should briefly summarize what is known in the general case for

$$\Delta z = f(x) - k(x)e^z \tag{5.44}$$

on (M^n, g). We will study equation (5.2) further in Chapter 7.

To repeat remarks made at the beginning of this Chapter, one can reduce to the case

$$\Delta u = c - he^u, \tag{5.45}$$

where c is a constant whose value is $\overline{f}$. Also, a necessary condition to be able to solve (5.45) is that in some open set $h(x)$ has the same sign as c (if $c = 0$, then this condition is that h changes sign, unless $h \equiv 0$). This equation is (5.1), except that there c and h had the *opposite* sign.

Multiplying (5.45) by e^{-u} and integrating over M (and integrating by parts) one finds that

$$\int_M h\,dx_g = c\int_M e^{-u}\,dx_g - \int_M e^{-u}|\nabla u|^2\,dx_g. \tag{5.46}$$

Thus, if $c \le 0$ then a second necessary condition is that $\int_M h\,dx_g < 0$ (unless $c = h \equiv 0$).

If $c = 0$ and $n = 2$, then, using the calculus of variations, one can show that these two conditions are necessary and sufficient to solve (5.45) [KW-1], but for $n \ge 3$, nothing more is known for this case $c = 0$.

If $c < 0$, then these two necessary conditions are *not* sufficient for a (5.45) to have a solution, even if $n = 1$ and $M = S^1$. We have seen that a sufficient condition is $h \le 0\,(\not\equiv 0)$; one can use sub and supersolution to prove [KW-1] that given any function h_0, if $h = h_0 + \alpha$, then there is a finite constant α_0 so that one can solve (5.45) if $\alpha < \alpha_0$ but not if $\alpha > \alpha_0$.

For $c > 0$ we have information only if $n = 2$. If $c > 0$ is sufficiently small, then using the calculus of variations one can solve (5.45) assuming only that h is positive somewhere on M. In the particular case of the sphere (S^2, g_0) with the standard metric, Moser [MJ-2] proved that if $c < 2$ then one can solve (5.45) if (and only if) h is positive somewhere. But if $c \ge 2$ then Kazdan-Warner [KW-1] found an *obstruction* to solvability. They proved that every solution must satisfy the identity

$$\int_{S^2} (\nabla h \cdot \nabla \varphi) e^u \, dx_g + (c-2) \int_{S^2} h \varphi e^u \, dx_g = 0, \tag{5.47}$$

where φ is any first order spherical harmonic (that is, φ is any solution of $-\Delta\varphi = 2\varphi$, so φ is any linear function $ax + by + cz$ or $\mathbb{R}^3$ restricted to S^2). In particular, if $c = 2$ then $h = \varphi +$ constant > 0 does not satisfy (5.47) so there is no solution of (5.45) in this case.

There has been further work on this (see [BE], [Au-4], [CGY] for references) but the situation is not at all clear. For $n \ge 3$ and $c > 0$ there is no information on any (M, g). In particular there is no known analog of the obstruction (5.47) to solving (5.44) for manifolds of dimension higher than two. Although in higher dimension (5.44) has no geometric significance, it is still surprising that we know nothing about it for the cases when $f(x) > 0$ that cause so much difficulty in dimension two. Note, however, there is a generalization of (5.45) to a complex Monge-Ampère equation on Kähler manifolds. We discuss this in Chapter 9.3 below.

Chapter 6

Implicit Function Theorem: Geometric Applications

6.1 Introduction

One basic procedure in attacking a problem is that if one can solve some special case—say by using symmetry—then one can often solve some cases that are near the special case. In physics this is called *perturbation theory*, while in mathematics one frequently calls it the *implicit function theorem*, and the related *bifurcation theory*. The implicit function theorem assumes some linearized map is invertible. Bifurcation theory (= the theory of singularities of maps) is used if this linearized map is *not* invertible.

Instead of discussing generalities, we will treat some specific applications. They can be read independently of each other.

6.2 Isothermal Coordinates

Let

$$g = ds^2 = E(x,y)\,dx^2 + 2F(x,y)\,dx\,dy + G(x,y)\,dy^2 \tag{6.1}$$

be a Riemannian metric on an open set in $\mathbb{R}^2$. If we make a change of coordinates $u = u(x,y)$ $v = v(x,y)$ then it is plausible that by a clever choice of the two functions u and v we can impose two conditions on g to simplify it. One standard choice is to arrange that in the new coordinates $E = G$ and $F = 0$, so

$$g = \lambda(u,v)(du^2 + dv^2) \tag{6.2}$$

for some positive function λ; these are called *isothermal coordinates*. We shall give two proofs that, *locally, one can always introduce isothermal coordinates*.

Proof 1. Use the Hodge $\star$ operator. On $\mathbb{R}^2$, at every point this is an isometry $\star : \Lambda^1 \to \Lambda^1$ sending a *1*-form α into an orthogonal *1*-form. In terms of the pointwise inner product of *1*-forms φ and ψ, the defining property of $\star$ is: $\varphi \wedge \star\psi = (\varphi, \psi)\,dx_g$. This implies that $\star$ is a pointwise isometry. The new coordinates are to have the properties $|du| = |dv|$ and $du \perp dv$. Thus, one seeks v as $dv = \star du$. This is an elliptic system to which the Local Solvability Theorem 4.3 applies. There are several ways of completing the details. One procedure, left as an exercise, is to use local coordinates as follows: beginning from (6.1), compute the Hodge $\star$ on 1-forms and use it to write out $dv = \star du$ as a first-order elliptic system for u and v. One refers to the equations $dv = \star du$ as the *Cauchy-Riemann equations* for the metric g.

As a slight alternate, if there is a solution of $dv = \star du$, then $d^2 v = 0$ implies that $d \star du = 0$, that is, $\Delta_g u = 0$, where Δ_g is the Laplacian in the given metric g. Once one has u, then v is found from $dv = \star du$ (note also $\Delta_g v = \star d \star dv = 0$). In addition to u satisfying $\Delta_g u = 0$, we also need the Jacobian of the map $(x,y) \mapsto (u,v)$ to be non-zero. Because $|du| = |dv|$, it is enough that $du \neq 0$. Thus, we seek a solution u of $\Delta_g u = 0$ with $du \neq 0$. By an easy explicit computation in local coordinates one can find constants a, b so that $u_0 = ax + by$ satisfies $\Delta_g u_0 = 0$ at the origin while $du_0 \neq 0$ there. Thus the local solvability Theorem 4.3 gives us the solvability of $\Delta_g u = 0$, $du \neq 0$ in some neighborhood of the origin. This completes the first proof.

Proof 2. For this proof, we use the fact that if a Riemannian metric g_1 is flat (in dimension two, this means the Gauss curvature is zero), then it is locally diffeomorphic to Euclidean space with its standard metric; the exponential map gives the diffeomorphism explicitly.[1] We will seek a function φ so that the *pointwise conformal metric* $g_1 = e^{2\varphi} g$ is flat, since then, as stated just above, for some diffeomorphism f we have $f^*(g_1) = du^2 + dv^2$. Thus $f^*(g) = e^{-2f^*(\varphi)}(du^2 + dv^2)$ and f is the desired change of coordinates. All we must do is to find φ. Using the standard formula (A.39) for the Gauss curvature K_1 of g_1 we see that φ should satisfy

$$\Delta_g \varphi = K - K_1 e^{2\varphi} = K, \tag{6.3}$$

where K is the Gauss curvature of g — and we used that $K_1 = 0$. As before, using explicit local coordinates, it is easy to find a quadratic polynomial

$$\varphi_0(x,y) = ax^2 + bxy + cy^2 \tag{6.4}$$

satisfying (6.3) at the origin; hence by the Local Solvability Theorem 4.3 there is a solution of (6.3) in some neighborhood of the origin.

In dimensions higher greater that two it is unclear what one should choose as the optimal local form for a Riemannian metric. DeTurck and Yang [DY] have shown that on a smooth *3*-manifold, one can always introduce local coordinates so that the metric is diagonalized. This problem is *not* elliptic.

6.3 Complex Structures

a) Complex Structures on $\mathbb{R}^2$

A *complex structure* is just a way to decide which functions are analytic. One customarily says $f \in C^1$ is analytic if $\partial f / \partial \bar{z} = 0$, that is,

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = 0. \tag{6.5}$$

How can we recognize these Cauchy-Riemann equations in other coordinates? In other words, say one is given two real vector fields

$$Q_j = a^j(x,y)\,\frac{\partial}{\partial x} + b^j(x,y)\,\frac{\partial}{\partial y}, \qquad j = 1, 2,$$

and let

$$Pf = (Q_1 + i\,Q_2)\,f.$$

[1] Riemann originally presented his curvature tensor precisely as the obstruction to finding a local change of coordinates to the standard Euclidean metric.

Can we find new coordinates $u = u(x, y)$, $v = v(x, y)$ so that in these new coordinates $Pf = 0$ is equivalent to $(\partial/\partial u + i\partial/\partial v)f = 0$?

A necessary condition is clearly that Q_1 and Q_2 be linearly independent. The Lewy example (4.13), which is not locally solvable, shows what can happen if Q_1 and Q_2 are dependent.

We claim this is also sufficient. Observe that if we have a solution $w = u + iv$ of $Pw = 0$ with ∇u and ∇v linearly independent, and if we use u and v as new coordinates, then by the chain rule, in these coordinates

$$P = \alpha(u,v)\,\frac{\partial}{\partial u} + \beta(u,v)\,\frac{\partial}{\partial v}$$

for some complex-valued functions α and β. But by substitution

$$0 = P(u + iv) = \alpha(u,v) + i\beta(u,v).$$

Thus $\alpha = -i\beta$ and $P = -i\beta(\partial/\partial u + i\partial/\partial v)$. This proves that $Pf = 0$ if and only if $(\partial/\partial u + i\partial/\partial v)f = 0$. The only gap is that we must locally solve $Pw = 0$. Since Q_1 and Q_2 are linearly independent, one can easily verify that $Pw = 0$ is elliptic so one obtains the local solvability with ∇u and ∇v independent by using the Local Solvability Theorem 4.3.

After some thought about complex structures, one can see that the results in this section are equivalent to our earlier discussion of isothermal coordinates.

b) Complex Structures on $\mathbb{R}^{2n}$

For several complex variables $z^1, \ldots, z^n$ one can similarly ask how one can recognize the Cauchy-Riemann equations

$$\partial f/\partial \bar{z}^1 = \partial f/\partial \bar{z}^2 = \cdots = \partial f/\partial \bar{z}^n = 0$$

in other coordinates. Now we are given n complex vector fields

$$P_j := \sum_{k=1}^{2n} a_{kj}\,\frac{\partial}{\partial x^k} = \sum_{k=1}^{n} c_{kj}\,\frac{\partial}{\partial z^k} + d_{kj}\,\frac{\partial}{\partial \bar{z}^k}, \qquad j = 1, \ldots, n \tag{6.6}$$

with $P_1, \ldots, P_n$, $\bar{P}_1, \ldots, \bar{P}_n$ linearly independent and seek a change of coordinates $\zeta = \phi(z, \bar{z})$ so that f satisfies $P_j f = 0$, $j = 1, \ldots, n$ if and only if $\partial f/\partial \bar{\zeta}^k = 0$, $k = 1, \ldots, n$. If we can find these new coordinates, then the P_j will be linear combinations of the $\partial/\partial \bar{\zeta}^k$. Consequently, a necessary condition is that

$$[P_j,\, P_k] = \text{ linear combination of } \{P_1, \ldots, P_n\}. \tag{6.7}$$

Newlander and Nirenberg (1957) proved that the linear independence and the integrability condition (6.7) are also sufficient that there are coordinates $\zeta^1, \ldots, \zeta^n$ so that $P_j f = 0$ are equivalent to the Cauchy-Riemann equations.

Just as in the simpler case of complex structures on $\mathbb{R}^2$, we will find solutions $\zeta^1, \ldots, \zeta^n$ of $P_j \zeta^k = 0$, $j, k = 1, \ldots, n$, with the gradients of the ζ^k's linearly independent. These will be the new coordinates.

We will sketch Malgrange's proof [Ma] of this result, following the exposition in [N-2]. Malgrange begins with the classical observation that the problem is solvable if the coefficients a_{jk} in (6.6) are real analytic since then one can obtain power series solutions (see [KN, Vol. 2, Appendix 1]); the integrability conditions (6.7) are formally just those of the Frobenius theorem. He solves the general case by showing there is a change of coordinates so that the equations are real analytic in the new coordinates. Then one can appeal to the real analytic case to complete the proof.

First a preliminary change of coordinates. If we freeze the coefficients in (6.6) at one point, say the origin, then for this constant coefficient system one can find a linear change of coordinates and solve the problem, that is, in these new coordinates $c_{jk} = \delta_{jk}$ and $d_{jk} = 0$. If me make this same linear change of coordinates in our variable coefficient case, then we obtain a system of the form (6.6) with $c_{jk}(0) = \delta_{jk}$ and $d_{jk}(0) = 0$. Since c_{jk} is now invertible near the origin, we can multiply by its inverse to rewrite (6.6) in the equivalent simpler form

$$P_j\zeta := \frac{\partial\zeta}{\partial\bar{z}^j} - \sum_k a_{kj}\,\frac{\partial\zeta}{\partial z^k} = 0, \qquad j = 1,\ldots,n, \tag{6.8}$$

with new coefficients a_{kj} and where we are thinking of $\zeta = (\zeta^1,\ldots,\zeta^n)$ as a complex vector. For short we write this as the matrix system

$$\frac{\partial\zeta}{\partial\bar{z}} = \frac{\partial\zeta}{\partial z}A, \quad \text{that is,} \quad \zeta_{\bar{z}} = \zeta_z\,A. \tag{6.9}$$

Because of the special form of (6.8), the commutators $[P_j, P_k]$ do not involve $\partial/\partial\bar{z}$. Thus the integrability conditions become simply

$$[P_j,\, P_k] = 0. \tag{6.10}$$

The key idea is to introduce new coordinates $w^j = w^j(z,\bar{z})$ in a clever way to be specified shortly, with $w_z(0) = I$, and $w_{\bar{z}}(0) = 0$. In these new coordinates (6.9) takes the form

$$\zeta_{\bar{w}} = \zeta_w B \tag{6.11}$$

where

$$B = (w_z A - w_{\bar{z}})(\bar{w}_{\bar{z}} - \bar{w}_z A)^{-1} \tag{6.12}$$

(note the condition on w at the origin ensures that $\bar{w}_{\bar{z}} - \bar{w}_z A$ is invertible near the origin). In these new coordinates the integrability conditions (6.10) for (6.11) take the form

$$\frac{\partial b_{ik}}{\partial\bar{w}^j} - \frac{\partial b_{ij}}{\partial\bar{w}^k} = \sum_r \left(b_{rj}\,\frac{\partial b_{ik}}{\partial w^r} - b_{rk}\,\frac{\partial b_{ij}}{\partial w^r} \right) \tag{6.13}$$

where we have written $B = (b_{ij})$.

For any choice of coordinates $w = w(z,\bar{z})$, the system (6.9)-(6.10) is entirely equivalent to (6.11), (6.13). Now we pick clever coordinates, requiring that they satisfy the additional conditions

$$\sum_k \frac{\partial b_{jk}}{\partial w^k} = 0, \qquad\qquad j = 1,\ldots,n. \tag{6.14}$$

It is not difficult to verify that these equations (6.13)-(6.14) for the coefficients b_{ij} as functions of the w^k are an overdetermined elliptic system with analytic coefficients. Therefore the functions b_{ij} are *analytic* functions of the w and $\bar{w}$. Consequently, we have reduced to the analytic case and conclude that the equations (6.11), (6.13) can be solved to give a solution with $\zeta_w(0) = I$, $\zeta_{\bar{w}}(0) = 0$.

It remains to be shown that the functions w^k can be found to satisfy (6.14). Using (6.12) and the chain rule (to express $\partial/\partial w$ in terms of $\partial/\partial z$ and $\partial/\partial\bar{z}$), the equations (6.14) become differential equations for w^k as functions of z and $\bar{z}$. Because A is zero to second order at the origin, this system is elliptic at the origin for the function $w \equiv z$; indeed, the linearized system is

$$-\sum_k \frac{\partial^2 w^j}{\partial z^k \partial \bar{z}^k} + \text{ lower order terms.}$$

Since $4\partial^2/\partial z^k \partial \bar{z}^k$ is just the Laplacian, the ellipticity is obvious. The local solvability of (6.14) for $w = w(z, \bar{z})$ with $w(0) = 0$, $w_z(0) = I$, $w_{\bar{z}}(0) = 0$ is now a consequence of the Local Solvability Theorem 4.3

6.4 Prescribing Gauss and Scalar Curvature

Let M be a compact n-dimensional manifold. Given a function S, is there a Riemannian metric g so that S is the scalar curvature of g?

If $n = 2$, then $S = 2K$, where K is the Gauss curvature. The Gauss-Bonnet theorem

$$\int_M K\, dA = 2\pi\chi(M), \tag{6.15}$$

where dA is the element of area and $\chi(M)$ is the Euler characteristic, gives an obvious necessary condition on K, namely, if $\chi(M) > 0$ then K must be positive somewhere, if $\chi(M) < 0$ then K must be negative somewhere, while if $\chi(M) = 0$ then K must change sign—unless it is identically zero.

For $n \geq 3$ there are other, more complex, topological restrictions that are not yet fully understood. There are topological obstructions to positive and to zero scalar curvature —for example, the torus T^n has a scalar curvature $S_g \geq 0$ if and only if g is flat, i.e. its sectional curvature is zero, while a *K-3* surface has no metric with $S_g > 0$ (see Chapter 7.2 some further remarks on this).

As a contrast, *every* compact M $(\dim M = n \geq 3)$ has a metric of negative scalar curvature (see Chapter 7.2) so there are no topological restrictions to negative scalar curvature. There are stronger results in two directions. First, in Corollary 7.3 using the theorem we will shortly prove, we will prove that for every compact manifold, any function that is negative somewhere is the scalar curvature of some metric. The second is the recent proof by Lohkamp [Lo] that every compact manifold of dimension at least three admits a smooth metric with negative Ricci curvature.

Let $S(g)$ denote the scalar curvature of g. It is computed using a complicated formula involving the derivatives of g up to order two (see (A.27)–(A.29)). Thus given our candidate, S, for the scalar curvature, we wish to solve the second order partial differential equation

$$S(g) = S. \tag{6.16}$$

This is one equation for the metric g, i.e. 1 equation for $\frac{1}{2}n(n+1)$ unknowns. We will show how to solve this equation under certain conditions. One key step is to observe that it is underdetermined elliptic.

Beginning with an arbitrary metric g_0, let $S_0 = S(g_0)$. The strategy in Step *1* is to use the implicit function theorem to solve $S(g) = S$ for all S near S_0, say $\|S - S_0\| < \epsilon$ in some appropriate norm (actually, to avoid degeneracies first one may have to perturb g_0 slightly). In Step *2* we seek a diffeomorphism φ such that $\|\varphi^*(S) - S_0\| < \epsilon$. If φ can be found, then, by Step *1* there is a metric g_1 such that $S(g_1) = \varphi^*(S)$. But for any metric $\varphi^* S(g) = S(\varphi^* g)$, because locally φ^* is just a change of coordinates. Therefore the metric $g = (\varphi^{-1})^*(g_1)$ satisfies $S(g) = S$. One additional flexibility we will use below is the scaling $S(cg) = c^{-1}S(g)$, where $c > 0$ is any constant.

Theorem 6.1 [Kazdan-Warner [KW, 4]] *Let* (M, g_0) *be a compact Riemannian manifold,* $\dim M \geq 2$ *, with* $S(g_0) = S_0$ *a constant. If* $S_0 \not\equiv 0$ *, then any function* S *having the same sign as* S_0 *somewhere is the scalar curvature of some metric, while if* $S_0 \equiv 0$ *, then any function* S *that changes sign is the scalar curvature of some metric.*

Proof. To avoid some mild technical complications, we will only carry out the details when $S_0 = -1$ (any negative constant would do as well). In Theorem 7.2 we will show that any compact M , $\dim M \geq 3$, has a metric with scalar curvature $S_0 \equiv -1$.

Step 1. Since $S(g_0) = S_0$, to use the implicit function theorem we need the linearization (or differential) of S at g_0 . This follows from the explicit formula (A.28) for the curvature. As in (A.32)–(A.34), here is the result in classical tensor notation:

$$Ah := S'(g_0)h = -\Delta_0 h^i{}_i + h^{ij}{}_{;ij} - h^{ij}(R_0)_{ij}, \tag{6.17}$$

where Δ_0 is the Laplacian and $R_0 = \mathrm{Ric}(g_0)$ the Ricci tensor, respectively, of g_0 , and h is a symmetric tensor. The covariant derivatives in (6.17) are in the g_0 metric. We compute the L^2 formal adjoint A^* in detail: for any smooth function u , the definition of A^* and integration by parts (twice) gives

$$\begin{aligned}\langle A^*u,\, h\rangle = \langle u,\, Ah\rangle &= \int [-u\Delta_0 h^i{}_i + uh^{ij}{}_{;ij} - uh^{ij}(R_0)_{ij}]\, dx_0 \\ &= \int [-(\Delta_0 u)h^i{}_i + u_{;ij}h^{ij} - u(R_0)_{ij}h^{ij}]\, dx_0 \\ &= \int [-(\Delta_0 u)(g_0)_{ij} + u_{;ij} - u(R_0)_{ij}]h^{ij}]\, dx_0,\end{aligned}$$

where dx_0 is the element of volume in the g_0 metric. Thus in coordinate-free notation

$$A^*u = -(\Delta_0 u)g_0 + \mathrm{Hess}_0(u) - u\,\mathrm{Ric}(g_0). \tag{6.18}$$

The principal symbol of A^* is (see Chapter 1.6)

$$[\sigma_\xi(A^*)z]_{ij} = (-|\xi|^2(g_0)_{ij} + \xi_i\xi_j)z. \tag{6.19}$$

This is injective for $\xi \neq 0$. (To see this, say z is in the kernel of the symbol. Take the trace of (6.19) and obtain $0 = (-n+1)|\xi|^2 z$; but $\xi \neq 0$, so $z = 0$.) Consequently the operator A^* is overdetermined elliptic. This implies that A is underdetermined elliptic, so A is underdetermined elliptic and AA^* is elliptic. We are thus led to seek our metric g in the special form $g = g_0 + A^*u$, that is, we solve the fourth order nonlinear elliptic equation $F(u) = S$, where

$$F(u) := S(g_0 + A^*u). \tag{6.20}$$

Note that this is one scalar equation for one unknown u . It is elliptic at $u = 0$ since $F'(0)v = S'(g_0)A^*v = AA^*v$, and we know AA^* is elliptic. Now $F(0) = S_0$. To apply the Perturbation Theorem 4.2 we need only check that for any f the linear equation $AA^*v = f$ has a unique solution u . By the Fredholm alternative (Theorem 2.4) the unique solvability of $AA^*v = f$ in various spaces is assured if $\ker AA^* = 0$. Thus, say $AA^*z = 0$. Then $A^*z = 0$ because in L^2 , $0 = \langle z,\, AA^*z\rangle = \|A^*z\|^2$. Taking the trace of (6.18) we obtain

$$-(n-1)\Delta z - S_0 z = 0. \tag{6.21}$$

Since $S_0 = -1 < 0$, either method of Example 2.9, or a direct application of the maximum principle shows that $z = 0$. Thus $\ker AA^* = 0$. For use in Step *2*, we will need L^p spaces, so we use the fact that $AA^* : H^{p,4} \to L^p$ is an isomorphism for any $1 < p < \infty$.

In addition we need the map F of (6.20) to be C^1 from $H^{p,4}$ to L^p. From the explicit formula for scalar curvature we see that F is quasilinear, that is, it has the form

$$F(u) = \sum_{|\alpha|=4} a^\alpha(x, \partial^\ell u)\partial^\alpha u + b(x, \partial^\ell u), \tag{6.22}$$

where $\ell \leq 3$. If we pick $p > n = \dim M$, then by the Sobolev Embedding Theorem 1.1, if $u_j \to u$ in $H^{p,4}$ then $u_j \to u$ uniformly in C^3; using this it is easy to see that if $p > n$ then F is a C^1 map from $H^{p,4}$ to L^p. By the implicit function theorem, F maps a neighborhood of zero in $H^{p,4}$ onto an L^p neighborhood of S_0. Thus, there is an $\epsilon > 0$ so that if

$$\|S - S_0\|_{L^p} < \epsilon, \tag{6.23}$$

then there is a solution $g = g_0 + A^*u$ of $S(g) = S$ and g will be sufficiently near g_0 to also be positive definite. Using elliptic regularity and a bootstrap argument as in Example 2.6 one can see that if $S \in C^\infty$ then $u \in C^\infty$ —just observe $u \in H^{p,4}$ for $p > n$ implies $u \in C^{3,\sigma}$ for some $\sigma > 0$ so the coefficients a^α and b_α in (6.22) are in $C^{3,\sigma}$, etc..

Step 2. We first observe the following obvious *approximation lemma*. Say $f : M \to \mathbb{R}$ is a continuous function and for some $x_0 \in M$ we have $f(x_0) = \gamma$. Then given any $1 < p < \infty$, there is a diffeomorphism $\varphi : M \to M$ so that $\varphi^* f$ is arbitrarily close to γ in L^p; in fact, pick φ so that a small neighborhood, $\mathcal{U}$, of x_0 is spread over most of M, and note that in $\mathcal{U}$ the function $f(x)$ is near $f(x_0) = \gamma$ (this type of approximation fails if we use the uniform norm).

Using this approximation lemma we now can complete the proof of the special case $S_0 = -1$. Since S is assumed negative somewhere, there is a point x_0 and a constant $c > 0$ so that $cS(x_0) = S_0 = -1$. With $\epsilon > 0$ from (6.23), pick a diffeomorphism φ so that $\|c\varphi^*(S) - S_0\|_{L^p} < \epsilon$. Then by Step *1* there is a solution g_1 of $S(g_1) = c\varphi^*(S)$, so the metric $g = (\varphi^{-1})^*(cg_1)$ is the desired solution of $S(g) = S$. $\square$

We have been fairly detailed in this proof so that one can see how the various parts of the theory are used. In the future we will usually delete the more routine steps. Note that in Step *1* above we used that $S_0 < 0$ only to make it easy to conclude that the linearization (6.21) is invertible. If $S_0 \geq 0$ this is not necessarily true (as on the standard round sphere S^2, or the flat torus); one then perturbs g_0 to make (6.21) invertible. Alas, this new S_0 is likely not a constant so one is forced to use a slightly more complicated version of the approximation lemma in Step *2* — since the version above assumes that $S_0 = \gamma$ is a constant.

Because every compact *2*-manifold has a metric with constant Gauss curvature, one consequence of Theorem 6.1 is that *a function $K \in C^\infty(M)$ is the Gauss curvature of a metric if and only if K satisfies the obvious Gauss-Bonnet sign condition* (see after (6.15)). The original proof of this [KW-2] used conformal deformation of the metric via equation (6.25) below.

In the 2-dimensional case it is elementary to solve an interesting related problem. For a 2-dimensional Riemannian manifold (M, g_0) with Gauss curvature K_0 and area element dA_0, the *curvature 2-form* Ω is

$$\Omega_0 = K_0 dA_0, \tag{6.24}$$

By Gauss-Bonnet (6.15), $\int_M \Omega_0 = 2\pi\chi(M)$. Conversely, given any 2-form Ω that satisfies this Gauss-Bonnet condition, is there a metric g so that Ω is the curvature 2-form for

g ? Wallach-Warner [WW] proved that the answer is "yes". Here is their proof. First fix some metric g_0 and seek a new metric *pointwise conformal* to g_0, that is, $g = e^{2w}g_0$ for some as yet unknown function w. Now for pointwise conformal metrics one has the formulas (see (A.35))

$$dA_g = e^{2w}dA_0 \qquad \text{and} \qquad K_g = (-\Delta_0 w + K_0)e^{-2w}, \tag{6.25}$$

where Δ_0 and K_0 are the Laplacian and Gauss curvature, respectively, of g_0. Thus

$$\Omega = (-\Delta_0 w + K_0)dA_0 = -\Delta_0 w\, dA_0 + \Omega_0$$

To realize a given Ω as Ω_g for some g we thus will seek a function w such that

$$-\Delta_0 w\, dA_0 = \Omega - \Omega_0 \tag{6.26}$$

Since $\int_M \Omega = 0$, by assumption, and $\int_M \Omega_0 = 0$, by Gauss-Bonnet, we can write $\Omega - \Omega_0 = f dA_0$ for some function f satisfying $\int_M f\, dA_0 = 0$. Thus, despite nonlinear expectations, (6.26) reduces to a simple linear equation

$$\Delta_0 w = f,$$

As we observed in Example 2.9, since $\int_M f\, dA_0 = 0$ this equation has a solution, which is unique except that we can always add any constant to w.

6.5 Prescribing the Ricci Tensor Locally

Next we investigate which tensors R_{ij} are locally Ricci tensors. Given a metric g, its Ricci curvature can be computed by a formula (A.27)–(A.28) involving the first two derivatives of g. We write this as $\operatorname{Ric}(g)$, and want to solve the partial differential equation

$$\operatorname{Ric}(g)_{ij} = R_{ij}. \tag{6.27}$$

Since g and R are both symmetric tensors, there are the same number of equations as unknowns; this makes us optimistic. However, Ric is invariant under the group of diffeomorphisms: for any diffeomorphism φ

$$\varphi^* \operatorname{Ric}(g) = \operatorname{Ric}(\varphi^* g). \tag{6.28}$$

Let φ_t be a family of diffeomorphisms with φ_0 =identity. Then, using (A.33) and the algebraic symmetries of the curvature tensor, the derivative of (6.28) with respect to t at $t = 0$ (see (A.33) yields the *second Bianchi identity*

$$0 = 2R^i{}_{k;i} - R^i{}_{i;k} = g^{ij}\left[\left(2\frac{\partial R_{ik}}{\partial x^j} - \frac{\partial R_{ij}}{\partial x^k}\right) - R^\ell{}_k\left(\frac{2\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right)\right], \tag{6.29}$$

where $k = 1, \ldots, n$. Details of this approach to the Bianchi identity are in [K-1][2]. These Bianchi identities are n additional conditions which g and R must satisfy so our initial optimism is gone. DeTurck [D-1] observed that the Bianchi identity is indeed an obstruction to solving (6.27) locally. One example where the Ricci equation (6.27) cannot be solved locally in $\mathbb{R}^n$, $n \geq 3$ is the following. Let

$$R = \begin{pmatrix} x^1 & 0 & \cdots & 0 \\ \vdots & & Q & \\ 0 & & & \end{pmatrix}, \tag{6.30}$$

[2]The proof of the first Bianchi identity in that paper is artificial, and should be ignored.

where $Q(x^2,\dots,x^n)$ is any $(n-1)\times(n-1)$ symmetric matrix whose elements do not depend on x^1 ($Q=0$ or $Q=I$ are fine). Then *there is no Riemannian metric satisfying the Bianchi identity* (6.29) *and hence no metric with Ricci curvature* R *in any neighborhood of the origin.* To see this, simply look at the case $k=1$ in the Bianchi identity (6.29) on the hyperplane $x^1=0$ to conclude that $g^{11}=0$ there; this is impossible for a positive definite metric.

DeTurck also proved the next result, that one can solve $\mathrm{Ric}(g)=R$ locally if R is invertible. (It is natural to guess that if, given R, there is a metric satisfying the Bianchi identity (6.29) then (6.27) is locally solvable. This is an open question.)

Theorem 6.2 [DeTurck [D-2]]. *Let R be an invertible symmetric tensor in a neighborhood of the origin in $\mathbb{R}^n$, $n\ge 3$. Then R is the Ricci tensor of some Riemannian metric in some neighborhood of the origin.*

Proof. If the equation (6.27) were elliptic, then one could try to apply the Local Solvability Theorem 4.3 (of course, this is doomed to fail by the non-existence example above). First of all, it is straightforward to see that (6.27) is solvable at the origin itself. For instance, one may find g_0 in the simple form $g_{ij}=[1+p(x)]\delta_{ij}$ conformal to the standard metric on $\mathbb{R}^n$, choosing $p(x)$ to be a homogeneous quadratic polynomial.

Let $\mathrm{Ric}'(g_0)$ be the linearization of the operator $\mathrm{Ric}(g)$ at the metric g_0 at the origin. From the explicit formula (A.33) for $\mathrm{Ric}'(g)$ one observes that in an orthonormal frame the principal symbol is

$$[\sigma_\xi(\mathrm{Ric}'(g_0))h]_{ij}=-\tfrac{1}{2}[h_{ij}|\xi|^2+\sum_s(h_{ss}\;\xi_i\;\xi_j-h_{is}\;\xi_s\xi_j-h_{js}\;\xi_s\;\xi_i)]. \tag{6.31}$$

Now σ_ξ maps symmetric matrices to symmetric matrices. It is routine to verify that the kernel consists precisely of the matrices of the form $h_{ij}=v_i\;\xi_j+v_j\;\xi_i$ for any covector v, that is, $h=\xi\otimes v+v\otimes\xi$. Thus σ_ξ is not an isomorphism so the equation (6.27) is not elliptic. Using (3.16) we see that these matrices h in $\ker\sigma_\xi$ are exactly those tangent to the orbit of the metric g_0 under the group of diffeomorphisms, i.e. $h=d(\varphi_t^*(g_0))/dt|_{t=0}$ where φ_t is a family of diffeomorphisms with $\varphi_0=\mathit{identity}$. This shows that the invariance (6.28) under the group of diffeomorphism is related to the non-ellipticity of the curvature equation (6.27).

Building on these observations, DeTurck found the following rather simple proof [D-5] of this theorem. To cope with the group of diffeomorphisms, he solves

$$\mathrm{Ric}(g)=\varphi^*(R), \tag{6.32}$$

where the unknowns are both g *and* the diffeomorphism φ (this idea was also used in the proof of Theorem 6.1). If one can solve (6.32), it is obvious that $g_1=(\varphi^{-1})^*g$ satisfies $\mathrm{Ric}(g_1)=R$, as desired.

Equation (6.32) has more unknowns than equations, so it is underdetermined. The n additional unknowns supplied by φ compensate for the n conditions imposed by the Bianchi identity. We shall shortly verify that (6.32) is elliptic if R is invertible. Since it is solvable at the origin—just use $\varphi_0=id.$ and g_0 from above—the local solvability will then follow from Theorem 4.3 along with the device from the proof of Corollary 2.5a —or from Section 6.4 above (see equation (6.20))—to pass from an underdetermined to a determined elliptic system.

One slight technical problem is that this equation involves second derivatives of g, but only first derivatives of φ. There are several ways to circumvent this. Following DeTurck (see [Be-2] for another variant of this) we seek the diffeomorphism, which we can locally write as $\varphi=(\varphi^1,\dots,\varphi^n)$, in the special form $\varphi^i=\sum_j\partial s^{ij}/\partial x^j$ for some (not necessarily symmetric) tensor s^{ij}. Then (6.32) becomes an equation for the symmetric tensor g and the tensor s. It involves the second derivatives of both. We compute the symbol of the right side of (6.32). First

$$\begin{aligned} T(s) := \varphi^*(R) &= \sum_{i,j,k,\ell} R_{ij}(\varphi(x)) \frac{\partial \varphi^i}{\partial x^k} \frac{\partial \varphi^j}{\partial x^\ell}\, dx^k\, dx^\ell \\ &= \sum R_{ij}(\varphi) \frac{\partial^2 s^{ip}}{\partial x^p \partial x^k} \frac{\partial^2 s^{jq}}{\partial x^q \partial x^\ell}\, dx^k\, dx^\ell. \end{aligned}$$

Because $\varphi_0 = id.$, that is, $\varphi_0^i(x) = x^i$, then $s_0^{ip} = \frac{1}{2}(x^i)^2 \delta^{ip}$ (no summation on the index i) so for any symmetric tensor τ the linearization of T at s_0 is

$$\begin{aligned} T'(s_0) := \varphi^*(R) &= \frac{\partial}{\partial \lambda} T(s_0 + \lambda\tau)\Big|_{\lambda=0} \\ &= \sum \Big(R_{i\ell} \frac{\partial^2 \tau^{ip}}{\partial x^p \partial x^k} + R_{kj} \frac{\partial^2 \tau^{jq}}{\partial x^p \partial x^\ell} \Big)\, dx^k\, dx^\ell + \cdots, \end{aligned}$$

where $\cdots$ represent terms having lower order derivatives of τ. Letting $R\tau$ be matrix multiplication, we then find the symbol is

$$[\sigma_\xi(T'(S_0))\tau]_{k\ell} = \sum_p [(R\tau)_{\ell p}\xi_p\xi_k + (R\tau)_{kp}\xi_p\xi_\ell]. \tag{6.33}$$

Consequently, the symbol of the linearization L of the operator $\mathrm{Ric}(g) - T(\sigma)$ at $(g_0,\, s_0)$ is

$$\sigma_\xi(L) \begin{pmatrix} h \\ \tau \end{pmatrix} := \sigma_\xi(\mathrm{Ric}'(g_0)h) - \sigma_\xi(T'(\sigma_0)\tau).$$

This map σ_ξ goes from the pair of matrices, h and τ (with h symmetric) to one symmetric matrix. Underdetermined ellipticity means that $\sigma_\xi(L)$ is surjective if $\xi \neq 0$; thus for any symmetric S one must solve

$$\sigma_\xi(L) \begin{pmatrix} h \\ \tau \end{pmatrix} = S. \tag{6.34}$$

The idea is to use the first term, $h_{ij}|\xi|^2$, in (6.31) to solve (6.34), and then pick τ so that the remaining terms cancel. Thus, let $h = -2S/|\xi|^2$. Using our assumption that R is invertible there is a τ so that $R\tau = -[2h - \mathrm{trace}(h)\, I]/4$. One can verify easily that this solves (6.34). Therefore (6.32) with $\varphi^i = \sum \partial s^{ij}/\partial x^j$, is underdetermined elliptic and locally solvable. □

6.6 Local Isometric Embedding of M^2 in $\mathbf{R}^3$ and $\mathbf{R}^4$

.

Let (M^2, g) be a two dimensional Riemannian manifold. When can one realize this, locally, as a small piece of a two dimensional surface in $\mathbb{R}^3$ or $\mathbb{R}^4$? We will prove that one can always locally isometrically embed in $\mathbb{R}^4$, and that one can embed in $\mathbb{R}^3$ in many cases—although, as we will discuss below, it is still unknown if one can always locally embed in $\mathbb{R}^3$.

Write

$$g = ds^2 = E(u,v)\, du^2 + 2F(u,v)\, du\, dv + G(u,v)\, dv^2.$$

The strategy to embed in $\mathbb{R}^3$ was first used by Weingarten [We]. He seeks a function $z(u,v)$ with $\nabla z(0) = 0$ so that the metric

$$g_1 = g - (dz)^2 \tag{6.35}$$

has zero Gauss curvature in some neighborhood of the origin. Then, just as in the second proof in our discussion of isothermal coordinates in Section 6.2, by using the exponential map we obtain new coordinates (x,y) so that $g_1 = dx^2 + dy^2$. Writing z in these new coordinates we obtain

$$g = g_1 + (dz)^2 = dx^2 + dy^2 + dz(x,y)^2. \tag{6.36}$$

Thus our metric g is exactly the metric induced on the surface $z = z(x,y)$ by its embedding in $\mathbb{R}^3$.

To carry out the details, let K and K_1 be the Gauss curvatures of g and $g_1 = g - (dz)^2$, respectively. We must find a function z so that $K_1 \equiv 0$ in a neighborhood of the origin. We can compute K_1 in terms of the derivatives of g_1 and hence of z. Write this operator as $K_1 = T(z)$ so we want to solve $T(z) = 0$. By a linear change of variables we may assume that in our coordinates $g_{ij} = \delta_{ij}$ at the origin. Then by an explicit computation (recall we want $\nabla z(0) = 0$)

$$T(z)|_0 = K_1(0) = K(0) - (z_{uu}z_{vv} - z_{uv}^2)\Big|_0. \tag{6.37}$$

If g is real analytic, then using the Cauchy-Kowalevsky theorem it is obvious that one can find an analytic solution of $T(z) = 0$ near the origin. This proves the existence of local isometric embedding in $\mathbb{R}^3$ for analytic metrics (Cartan [C] and Janet [J] extended this to all dimensions. (see also [Sp])).

One can also prove the *local isometric embedding in* $\mathbb{R}^3$ *if* $K(0) > 0$ (and $K \in C^\infty$) since then $z_0(u,v) = \frac{1}{2}(u^2+v^2)\sqrt{K(0)}$ is a solution of $T(z_0) = 0$ at the origin and the equation is elliptic there (Chapter 2.2); consequently one can apply the Local Solvability Theorem 4.3 for elliptic equations.

In addition, it turns out that if $K(0) < 0$ then the equation $T(z) = 0$ is locally solvable (and hence one can locally embed in $\mathbb{R}^3$) since then the equation $T(z) = 0$ is hyperbolic, but we have not discussed the machinery here. Thus, the only remaining unresolved case is when $K(0) = 0$ when the equation $T(z) = 0$ is neither elliptic nor hyperbolic. This case is partially treated in the work of [Lin-1, Lin-2], who obtained the embedding if either $K \geq 0$ near the origin or if $K(0) = 0$ but $\nabla K(0) \neq 0$. It is essentially the Monge-Ampère equation discussed in the Example at the end of Section 4.5,

It is quite easy to prove that *one can always locally isometrically embed* (M^2, g) *in* $\mathbb{R}^4$. There are several proofs of this. Our first—and shortest—proof uses the observation from (6.37) that if $w(u,v) = cuv$, then by choosing c large the metric $g_1 = g - dw^2$ has positive curvature in a neighborhood of the origin. Thus, as proved above, we can isometrically embed g_1 in $\mathbb{R}^3$, so

$$g_1 = dx^2 + dy^2 + dz(x,y)^2$$

for some function $z(x,y)$. Write $w(u,v)$ in terms of these new coordinates (x,y) to conclude that

$$g = g_1 + dw^2 = dx^2 + dy^2 + dz(x,y)^2 + dw(x,y)^2,$$

which gives the desired embedding in $\mathbb{R}^4$ with coordinates $(x,\, y,\, z,\, w)$..

The second proof (due to Poznyak [P]) that one can always locally embed in $\mathbb{R}^4$ is longer, but it yields more, showing that one can embed (M^2, g) on the special three dimensional surface $\Sigma^3 \hookrightarrow \mathbb{R}^4$ defined by

$$(\epsilon\rho\cos\frac{\theta}{\epsilon}, \epsilon\rho\sin\frac{\theta}{\epsilon}, \epsilon\rho\cos\frac{\varphi}{\epsilon}, \epsilon\rho\sin\frac{\varphi}{\epsilon}),$$

where $\epsilon > 0$ is a parameter. The metric on Σ^3 is

$$g_0 = \rho^2(d\theta^2 + d\varphi^2) + 2\epsilon^2 d\rho^2. \tag{6.38}$$

Write the given metric g in isothermal coordinates (see Section 6.2)

$$g = \lambda^2(u,v)(du^2 + dv^2). \tag{6.39}$$

We seek functions $\rho = e^{w(u,v)}$, $\theta = \theta(u,v)$, $\varphi = \varphi(u,v)$ so that $g = g_0$; then equations (6.38)–(6.39) give

$$\gamma := \lambda^2 e^{-2w}(du^2 + dv^2) - 2\epsilon^2 dw^2 = d\theta^2 + d\varphi^2.$$

If we regard γ as a metric, then, just as before, it is enough to show that for some function $w(u,v)$ the metric γ is flat, that is, the Gauss curvature K_γ of γ zero. Now from (A.39) the Gauss curvature K_1 of $g_1 := \lambda^2 e^{-2w}(du^2 + dv^2)$ is

$$K_1 = [\Delta w - \Delta(\log\,\lambda)]\lambda^{-2}e^{2w},$$

where $\Delta w = w_{uu} + w_{vv}$. Since $\gamma = g_1 - 2\epsilon^2 dw^2$, from (6.37) at the origin we have

$$K_\gamma = [\Delta w - \Delta(\log\,\lambda)]\lambda^{-2}e^{2w} - 2\epsilon^2(w_{uu}w_{vv} - w_{uv}^2).$$

We want to solve the equation $K_\gamma = 0$ for w in some neighborhood of the origin for some $\epsilon > 0$. One can clearly solve this equation if $\epsilon = 0$: a solution is $w(u,v) = au + bv + \log\,\lambda(u,v)$, where a and b are constants chosen so $\nabla w(0,0) = 0$. By the implicit function theorem it is straightforward to conclude that one can solve this elliptic equation $K_\gamma = 0$ for w as long as ϵ is sufficiently small.

6.7 Bifurcation Theory

Even if the implicit/inverse function theorem is not applicable, one can often get some valuable information. Say one wants to solve the scalar equation $T(x) = y$. Assume that $T(x_0) = y_0$ and seek a solution y near y_0. Formally, one can use a Taylor series

$$T(x) = T(x_0) + T'(x_0)(x - x_0) + \tfrac{1}{2}T''(x_0)(x - x_0)^2 + \cdots .$$

If $T'(x_0)$ is not invertible, then one can not apply the inverse function theorem, and a deeper analysis is needed—using $T''(x_0)$ and possibly higher order derivatives. The simplest example is the map $f(x) = x^3$, which is bijective, even though $f'(0) = 0$. For the higher dimensional case where $x = (x_1, \ldots, x_n)$, the Morse lemma, which describes a smooth real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ near a non-degenerate critical point, gives a simple criterion for solving the scalar equation $T(x) = y$. The Morse lemma states there are new coordinates so that near the critical point

$$f(x) = f(0) + (x_1^2 + \cdots + x_k^2) - (x_{k+1}^2 + \cdots + x_n^2).$$

From this it is evident that one can solve $f(x) = y$ for all y near $y_0 = f(0)$ if and only if $k \neq 0$ or n, since then $f(x) - f(0)$ changes sign for small x. The study of singularities of maps between finite dimensional manifolds is especially useful for elliptic differential equations $T(u) = f$ since ellipticity allows one to reduce to the finite dimensional case. There is a good introductory exposition in [N-3, Chapter 2.7 and Chapter 3]. All we will do here is give two specific examples which may be useful to keep in mind while reading the general theory.

EXAMPLE 6.1 The first example is solving

$$F(u, h) := -\Delta u + c - he^u = 0 \tag{6.40}$$

on a compact manifold M^2, where c is a given function. Given $h(x)$ we want a solution u. Clearly $F(0, c) = 0$ and we seek a solution for h near c. The linearization at $u = 0$, $h = c$, is

$$Lv = F_u(0, c)v = -\Delta v - cv.$$

Since L is self-adjoint, if $\ker L = 0$ then by the Fredholm alternative (Theorem 2.4) L is a bijection from $C^{2,\alpha}(M) \to C^\alpha(M)$ and also, from $H^{p,2}(M^n) \to L^p(M^n)$. For F to be a C^1 map we need $p > n/2$ (since then $H^{p,2} \hookrightarrow C^0$). Thus, if $\ker L = 0$ then given any h near c we can solve $F(u, h) = 0$. However there is trouble if $\ker L = 0$ because then L is not a bijection so the implicit function theorem can not be applied. It is an instructive exercise to work out a local (near $u = 0$) description of when one can solve (6.40) in the case $c = 0$, so $\ker L = \operatorname{coker} L$ are one dimensional and are spanned by the constant function, and also the case $c = 2$ on (S^2, g_0) — where g_0 is the standard metric—so $\ker L = \operatorname{coker} L$ are both three dimensional and are spanned by the first order spherical harmonics. □

EXAMPLE 6.2 A second instructive example is, on compact M, finding nontrivial solutions $(u \not\equiv 0)$ of

$$F(u, \lambda) := \Delta u + \lambda \sinh u = 0. \tag{6.41}$$

The existence of nontrivial solutions will depend on λ. For instance if $\lambda = 0$, then $u =$ *const.* is a solution.

By an easy argument [or just use part *b*) of the Comparison Theorem 4.4], one sees that if $\lambda < 0$ then the only solution of (6.41) is $u = 0$. To find other solutions we consider the linearized equation

$$Lv = F_u(0, \lambda)v = \Delta v + \lambda v.$$

If λ is *not* an eigenvalue of the Laplacian, then by the implicit function theorem the only solution of $F(u, \lambda) = 0$ near $u = 0$ is the zero solution itself. Thus, to find a non-trivial solution we let $\lambda = \lambda_0$ be an eigenvalue of the Laplacian. If the eigenspace is one dimensional, one can show that there is a non-trivial solution of (6.41) for all λ near λ_0 (see Theorem (3.22) and page 79 in [N-3]). □

Chapter 7

Scalar Curvature

[*not yet revised*]

7.1 Introduction

Because the scalar curvature is such a weak invariant of the metric (one averages the curvature *twice* to get the scalar curvature), it is not at all clear that there are any topological obstructions to functions being scalar curvatures, except in dimension two where we have Gauss-Bonnet. After discussing some obstructions—and techniques for finding them—we consider the question of prescribing the scalar curvature. The most naive way of deforming a metric is pointwise conformally: $g_1 = pg$ for some positive function p. In 1960 Yamabe [Ym] asked if one can find a function p so that g_1 has constant scalar curvature; he viewed this as a first step in finding Einstein metrics (see Chapter 9 below). This would then give a proof of the Poincaré conjecture in dimension three. Yamabe's work had a serious error and problem is still unsolved, although we now have some good information, which we present below. The reader may find the survey lecture [BB] useful.

7.2 Topological Obstructions

As we mentioned in Chapter 3.6, Lichnerowicz [Li] used the Bochner technique to find the first topological obstructions to metrics with positive scalar curvature. Hitchin later extended Lichnerowicz's argument to show that certain exotic spheres do not have positive scalar curvature metrics. There has been a striking generalization by Gromov-Lawson (see the discussion in [LM]). One of their results is that if a compact M has a metric with non-positive sectional curvature, then there is no metric with positive scalar curvature. In particular, the torus T^n has no metric with positive scalar curvature. The same ideas, first noticed by Kazdan-Warner [KW-5] show that some manifolds have topological obstructions to zero scalar curvature metrics.

Schoen-Yau [SY-1] used a totally different method to find topological obstructions to positive scalar curvature metrics. They viewed minimal surfaces as the analog of geodesics. Just as the second variation of arc length leads one to the Jacobi equation, which then gives important relationships between curvature and geodesics, the second variation of surface area gives useful information. The most difficult step, by Sachs-Uhlenbeck [SaU-1, 2] and Schoen-Yau [SY-1], is proving the existence of a minimal submanifold of a given manifold. This is done under topological assumptions analogous to those used by Synge in his results for geodesics. Then the second variation formula is utilized.

Schoen-Yau [SY-2] later extended their technique to prove the positive mass conjecture in general relativity (this problem is very closely related to positive scalar curvature). Subsequently, Witten [Wi] gave a different proof of the positive mass conjecture using harmonic spinors and the Dirac operator (for additional references, see the survey lecture [K-2]).

Since spinors and the Dirac operator are so closely related to positive scalar curvature, it would be valuable if one could find some way to use them to deform metrics in an intelligent way, augmenting the simple pointwise conformal deformations.

So far, we have mentioned obstructions to positive and to zero scalar curvature. What about negative scalar curvature? It turns out there are none, as was first observed by Aubin [Au-3] extending earlier work of Avez. In fact, as we mentioned in Chapter 6.4 there are no obstructions to negative Ricci curvature (see the discussion in Chapter 9). The key step in Aubin's construction is the following.

Lemma 7.1 *Let M^n, $n \geq 3$, be a compact manifold. There is a metric g with negative total scalar curvature, that is, $\int_M S_g \, dx_g < 0$.*

For the proof, one fixes a metric g_0 on M and then deforms it only in a very small set to achieve negative total scalar curvature. The simplest deformation is due to Bérard-Bergery and can be found in [Be-2]. One uses this for the next result.

Theorem 7.2 *Let (M^n, g) be a compact Riemannian manifold with negative total scalar curvature. Then there is a pointwise conformal metric, $g_1 = pg$ with $p > 0$, having constant negative scalar curvature, $S_1 = -1$.*

Proof. For amusement, we first do the classical case $n = 2$. Seek g_1 as $g_1 = e^u g$. Thus we wish to solve (5.4), namely $\Delta u = S(x) + e^u$, where $\int_M S \, dx_g < 0$. This is an immediate consequence of Theorem 5.8 — or else one can use the procedure used for (5.3) to reduce the problem to (5.45), and then apply any of the techniques of Chapter 5.

For $n \geq 3$ as in Chapter 5.1 we seek g_1 as

$$g_1 = u^{4/(n-2)} g, \tag{7.1}$$

and consequently must solve (5.6):

$$L_g u := -\gamma \Delta_g u + S_g u = -u^\alpha, \tag{7.2}$$

where $\gamma = 4(n-1)/(n-2)$ and $\alpha = (n+2)/(n-2) > 1$. Following [KW-5], let $\lambda_1(g)$ be the lowest eigenvalue of the linear operator L_g defined by the left side of (7.2) and let $\varphi > 0$ by a corresponding eigenfunction (here we used Proposition 5.3). Using $\psi \equiv 1$ in the Rayleigh quotient (5.18), the assumption on S implies that $\lambda_1 < 0$. Now we solve (7.2) using sub- and supersolutions (Chapter 5.5) of the form $u_\pm = c_\pm \varphi$, where $0 < c_- < c_+$ are constants. For example we want $L_g u_+ \geq -u_+^\alpha$, that is, $\lambda_1 c\varphi \geq -(c\varphi)^\alpha$ so $0 < -\lambda_1 < (c\varphi)^{\alpha-1}$ which is clearly satisfied for all large c. □

As a slight variant of the above proof, we should have proceeded in two steps, first using the eigenfunction φ to define the conformal metric $g' = \varphi^{4/(n-2)} g$. From (5.6) this has everywhere negative scalar curvature.

$$S_{g'} = \lambda_1(g) \varphi^{1-(n+2)/(n-2)}. \tag{7.3}$$

For the second step we seek $g_1 = u^{4/(n-2)} g'$, and thus solve (7.2) using g' instead of g as the metric on the left side. But since $S_{g'} < 0$, this equation is now in the form used in Chapter 5, so any of those techniques can be used.

The next Corollary follows immediately from Lemma 7.1, Theorem 7.2, and Theorem 6.1.

Corollary 7.3 [KAZDAN-WARNER] *On a compact manifold* (M^n, g) *, any function that is negative somewhere is the scalar curvature of some metric.*

In theorem the prescribed function need not be very smooth, even L^p for some $p > n$ is adequate and gives a metric in $H^{p,2}$. But the smoother the function,the smoother the metric.

The formula (7.3) shows how, given any metric, one can use the eigenfunction $\varphi > 0$ to find a pointwise conformal metric whose curvature has (everywhere) the same sign as the eigenvalue λ_1 . In particular, if $S_g \geq 0 (\not\equiv 0)$ there is a conformal metric g_1 with $S_1 > 0$, and similarly if $S_g \leq 0 (\not\equiv 0)$.

Kazdan-Warner [KW-5] have shown how to use the functional $\lambda_1(g)$ and eigenfunction φ to obtain a number of results, such as the following which points out that if one has a positive scalar curvature metric, then there is a zero scalar curvature metric. The torus, T^n , shows that the converse is not true.

Proposition 7.4 *If* M^n *,* $n \geq 3$ *, has a metric of positive scalar curvature, then it has one with zero scalar curvature.*

Proof. Let g_0 be the given metric with $S_{g_0} \geq 0$ and let g_1 be the metric of Theorem 7.2 (or of Lemma 7.1) with $\lambda_1(g_1) < 0$. Consider the metrics $g_t = tg_1 + (1-t)g_0$ and corresponding lowest eigenvalue $\lambda_1(g_t)$ of L_{g_t} defined by (7.2). Now $\lambda_1(g_t)$ is a continuous function of t (in fact, it is a real analytic function of t [Ka]) with $\lambda_1(g_0) > 0$ and $\lambda_1(g_1) < 0$. Thus for some $0 < t < 1$, $\lambda_1(g_t) = 0$ so by (7.3) $S_{g_t} = 0$, as desired. □

7.3 The Yamabe Problem, Analytic Part.

Given a metric g on M^n , $n \geq 3$, Yamabe [Ym] asked if there is a pointwise conformal metric $g_1 = u^{4/(n-2)} g$, $u > 0$, having constant scalar curvature. Thus, one wants to find a positive solution, $u > 0$, of

$$Lu := -\gamma \Delta u + Su = ku^\alpha, \tag{7.4}$$

where $\gamma = 4(n-1)/(n-2)$, and $\alpha = (n+2)/(n-2)$, and k is some constant. It is easy to see that the constant k has the same sign as the lowest eigenvalue, λ_1 , of L : just take the $L_2(M)$ inner product of (7.4) with the eigenfunction $\varphi > 0$ to obtain

$$\lambda_1 \langle \varphi,\, u \rangle = \langle L\varphi,\, u \rangle = \langle \varphi,\, Lu \rangle = k \int_M \varphi u^\alpha \, dx_g.$$

If $\lambda_1 = 0$, then $u = \varphi$ is an obvious solution of (7.4) with $k = 0$, while if $\lambda_1 < 0$, then Theorem 7.2 showed how one can solve the problem; moreover, the precise value of the exponent α was unimportant since all we used was $\alpha > 1$.

The case $\lambda_1 > 0$ is much more difficult (in fact, past experience has shown that positive curvature is usually more difficult than negative curvature). Here the precise value of α becomes quite significant. We will use the calculus of variations and follow the procedure used to find the lowest eigenvalue of $Lu = \lambda_1 u$ in Proposition 5.3. The procedure is clearer if we eliminate some of the geometry and consider the equation

$$-\Delta u + cu = f|u|^\alpha, \qquad \alpha = (n+2)/(n-2), \tag{7.5}$$

where c and $f > 0$ are given function. For convenience we may scale the metric (replace g by (const.) g) to have $\mathrm{Vol}(M, g) = 1$. Just as in (5.18) we seek a minimum of the functional

$$J(u) = \frac{\int_M (|\nabla u|^2 + cu^2)\, dx_g}{(\int_M f|u|^N\, dx_g)^{2/N}}, \qquad u > 0, \tag{7.6}$$

where $N = \alpha + 1 = 2n/(n-2)$. Multiplying u by a positive constant, minimizing J in (7.6) on $H^{2,1}(M)$ is equivalent to minimizing the functional

$$J(u) = \int_M (|\nabla u|^2 + cu^2)\, dx_g \tag{7.7}$$

on the set

$$Q = \{u \in H^{2,1}(M) : \quad \int_M f|u|^N\, dx_g = 1\} \tag{7.8}$$

(in geometric terms, if $g_1 = u^{4/(n-2)}g$, then $dx_{g_1} = u^N dx_g$ so the condition $\int fu^N\, dx_g = 1$ simply normalizes the volume of the metric g_1).

Hölder's inequality tells us that J is bounded below on Q because

$$\begin{aligned} \Big|\int_M cu^2\, dx_g\Big| &\leq \max|c(x)| \int_M u^2\, dx_g \\ &\leq \text{const.}\Big(\int_M |u|^N\, dx_g\Big)^{2/N} \leq \text{const.}\Big(\int_M f|u|^N\, dx_g\Big)^{2/N}. \end{aligned}$$

Let

$$\sigma = \inf_{\epsilon A} J(u), \tag{7.9}$$

and let $u_j \in Q$ be a minimizing sequence. Then

$$\int_M (|\nabla u_j|^2 + u_j^2)\, dx_g = J(u_j) + \int_M (1-c)u_j^2\, dx_g \leq \text{ constant}$$

so the sequence $\{u_j\}$ is in a bounded set in $H^{2,1}(M)$. Consequently there is a weakly convergent subsequence, which we relabel u_j, with $u_j \rightharpoonup u$ in $H^{2,1}(M)$. The difficulty is showing that $\int_M fu^N\, dx_g = 1$. By the compactness of the Sobolev Embedding Theorem 1.1, this sequence converges strongly $u_j \to u$ in L^p for all $p < 2n/(n-2) = N$, but not for $p = N$, which is what we need to prove $u \in Q$. Thus, if we replace the constant N by $N - \epsilon$ (and equivalently, α in (7.5) by $\alpha - \epsilon$), then one can continue to imitate the proof of Proposition 5.3 and prove the existence of a positive solution of (7.5) for any exponent α less that $(n+2)/(n-2)$. (At the final step one may have to replace u by (constant) u to eliminate the Lagrange multiplier.) However the compactness of the embedding $H^{2,1} \hookrightarrow L^p$ fails precisely at the case $p = 2n/(n-2)$ of geometric interest (equation (7.5) —with the same difficult exponent—also arises in the study of Yang-Mills fields).

At this stage, it is not at all clear if the difficulty we are encountering is because of an inefficient method or because of some genuine obstruction. For instance, if $c < 0$ and $f < 0$ then we would still have the above difficulties, while the methods of Chapter 5 give many ways of solving (7.5), with the value of $\alpha > 1$ being irrelevant. In Theorem 7.7 below we will show that the trouble we are having is genuine and basic to the problem, not a defect of the method.

We must work harder to obtain results here. Let $\Lambda_n > 0$ be the best constant in the Sobolev embedding of $H^{2,1}(\mathbb{R}^n) \hookrightarrow L^N(\mathbb{R}^n)$, with $N = 2n/(n-2)$, as above. Then

$$\Lambda_n = \min \frac{\int_M |\nabla \varphi|^2\, dx_g}{(\int_M |\varphi|^N\, dx_g)^{2/N}} \qquad (M = \mathbb{R}^n) \tag{7.10}$$

for all $\varphi \in H^{2,1}(\mathbb{R}^n)$. It turns out that $\Lambda_n = \frac{n(n-2)}{4}\omega_n^{2/n}$, where ω_n is the volume of the standard sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$ and that the same constant is optimal for any compact (M^n, g), independent of M^n or the metric g (see [Au-4, Theorem 2.30]): given any $\epsilon > 0$ there is a constant A_ϵ such that for all $\varphi \in H^{2,1}(M^n)$

$$\Lambda_n \|\varphi\|_{L^N}^2 \leq (1+\epsilon)\|\nabla\varphi\|_{L^2}^2 + A_\epsilon \|\varphi\|_{L^2}^2. \tag{7.11}$$

Comparing the variational problems (7.10) and (7.6), it is plausible that Λ_n and σ (see (7.9)) are related, but the following result of Aubin shows these constants are very closely related. Our proof uses some ideas from [BN].

Theorem 7.5 *(Aubin, see [Au-4]) Let (M^n, g) be a compact Riemannian manifold and define σ and Λ_n as above. Then*

$$\sigma \leq \Lambda_n (\max f)^{-2/N}. \tag{7.12}$$

Moreover, if strict inequality holds in (7.12), *then there is a function $u \in Q$ minimizing J.*

Proof. For a sphere of radius ρ, $S^n(\rho) \hookrightarrow \mathbb{R}^{n+1}$, the Mobius transformations give us many conformal metrics with constant curvature. Their scalar curvature is

$$S = n(n-1)/\rho^2. \tag{7.13}$$

The Mobius transformations we use are those induced on $S^n(\rho)$ under stereographic projection from S^n by the map $z \to tz$ on $\mathbb{R}^n$.
The induced metric on $S^n(\rho)$ is

$$g_t = \left[\frac{2t}{t^2+1+(t^2-1)\cos(r/\rho)}\right]^2 g_1, \tag{7.14}$$

where g_1 is the standard metric on $S^n(\rho)$ and $r = d(x, x_0)$ is the distance (arc length) from a point $x \in S^n(\rho)$ to the fixed point, x_0, of this Mobius transformation. Then writing $g_t = \psi_t^{4/(n-2)} g_1$, as in (7.1), from (7.14) we find that

$$\psi_t(x) = \left[\frac{2t}{t^2+1+(t^2-1)\cos(r/\rho)}\right]^{(n-2)/2} \tag{7.15}$$

and from (5.6), (7.13)

$$-\frac{4(n-1)}{n-2}\Delta\psi_t + \frac{n(n-1)}{\rho^2}\psi_t = \frac{n(n-1)}{\rho^2}\psi_t^{(n+2)/(n-2)}$$

that is,

$$-\Delta\psi_t + \frac{n(n-2)}{4\rho^2}\psi_t = \frac{n(n-2)}{4\rho^2}\psi_t^{(n+2)/(n-2)}. \tag{7.16}$$

The first step is to modify these special functions so they can be used in $J(\psi_t)$ for any manifold M^n. Pick $0 < \delta <$ injectivity radius of (M^n, g) and let $\eta \in C^\infty(\mathbb{R})$ satisfy $\eta(s) = 1$ for $s < \delta/2$, $\eta(s) = 0$ for $s > \delta$. For any point $x_0 \in M$, let $r = d(x, x_0)$ and let

$$\varphi_t(r) = \eta(r)\psi_t(r) \in C^\infty(M). \tag{7.17}$$

A long computation reveals that for $t \to 0$ we have

$$J(\varphi_t) = \Lambda_n f(x_0)^{-2/N} + o(t). \tag{7.18}$$

By picking x_0 at the point where f has its maximum, the right side is minimized. Since $\sigma = \inf J$, this proves (7.12).

To prove the last sentence in the theorem, note that before (7.9) we proved J is bounded below by σ so let $u_j \in Q$ satisfy $J(u_j) \downarrow \sigma$. These u_j are in a bounded set in $H^{2,1}(M)$ so by the Sobolev theorem there is a subsequence, which we relabel u_j, satisfying

$$\begin{aligned} &u_j \rightharpoonup u \quad \text{weakly in } H^{2,1}(M) \\ &u_j \rightharpoonup u \quad \text{in } L^2(M) \text{ and almost everywhere.} \end{aligned}$$

Let $v_j = u_j - u$ so $v_j \rightharpoonup 0$ in $H^{2,1}$ and $v_j \to 0$ a.e. in L^p for all $p < N$. We will be done if we can show that $v_j \to 0$ strongly in $H^{2,1}$ because then, by the Sobolev Theorem, $v_j \to 0$ strongly in L^N so $u_j \to u$ strongly in L^N. This implies that $u \in Q$ and $J(u) = \sigma$, that is, u gives the desired minimum.

To estimate v_j we first estimate the numerator in (7.6). Now because $v_j \rightharpoonup 0$ in $H^{2,1}(M)$:

$$\begin{aligned} \int \left(|\nabla u_j|^2 + c u_j^2\right) dx_g &= \int \left(|\nabla(u+v_j)|^2 + c(u+v_j)^2\right) dx_g \\ &= \int \left(|\nabla u|^2 + c u^2\right) dx + \int |\nabla v_j|^2 \, dx_g + o(1) \end{aligned}$$

so by the definition of σ

$$\geq \sigma(\int f|u|^N \, dx_g)^{2/N} + \int |\nabla v_j|^2 \, dx_g + o(1). \tag{7.19}$$

Next we estimate the denominator in (7.6). For this we need the observation (see [BL]) that

$$\int f|u_j|^N \, dx_g = \int f|u+v_j|^N \, dx_g = \int f|u|^N \, dx_g + \int f|v_j|^N \, dx_g + o(1).$$

Combined with the Sobolev Inequality (7.11) and $v_j \to 0$ in L^2, we find

$$\begin{aligned} 1 =& \Big(\int f|u_j|^N \, dx_g\Big)^{2/N} \\ \leq& \Big(\int f|u|^N \, dx_g\Big)^{2/N} + \frac{(\max f)}{\Lambda_n}^{2/N} (1+\epsilon) \int |\nabla v_j|^2 \, dx_g + o(1) \end{aligned} \tag{7.20}$$

Because $J(u_j) = \sigma + o(1)$, substituting (7.19) and (7.20) into (7.6) we obtain

$$\sigma\Big(\int f|u|^N \, dx_g\Big)^{2/N} + \int |\nabla v_j|^2 \, dx_g + o(1)$$

$$\leq [\sigma + o(1)]\Big\{\Big(\int f|u|^N \, dx_g\Big)^{2/N} + \frac{(\max f)^{2/N}}{\Lambda_n}(1+\epsilon) \int |\nabla v_j|^2 \, dx_g\Big\} + o(1).$$

Thus,

$$\left(1 - \frac{(1+\epsilon)\sigma(\max f)^{2/N}}{\Lambda_n}\right) \int |\nabla v_j|^2 \, dx_g \leq o(1). \tag{7.21}$$

Since strict inequality holds in (7.12), we can pick some $\epsilon > 0$ so that the leading coefficient is positive. Consequently $\int |\nabla v_j|^2 \, dx_g \to 0$ and hence $v_j \to 0$ in $H^{2,1}(M)$. $\square$

Once we have a minimum (or any critical point) of J, we can obtain a smooth positive solution of (7.5).

Corollary 7.6 *Let* $c > 0$, $f > 0$ *be smooth functions. If* (7.6) *has a critical point in* $H^{2,1}$, *in particular if* (7.12) *holds, then there is a solution* $0 < u \in C^\infty(M)$ *of* (7.5).

Proof. Let $w \in H^{2,1}(M)$ be a critical point of J. Then $|w|$ is also in $H^{2,1}(M)$ and $|\nabla w| = |\nabla(|w|)|$ almost everywhere ([Au-4, p. 82]). Therefore $J(w) = J(|w|)$ so $u = |w| \geq 0$ is also a critical point. Consequently, for any $z \in H^{2,1}(M)$

$$\int (\nabla u \cdot \nabla z + cuz - \beta f u^\alpha z)\, dx_g = 0, \tag{7.22}$$

where β is an (unknown) Lagrange multiplier. To prove that u is smooth, one can use standard elliptic regularity (see the end of Theorem 5.2 if one knows that $u \in L^p$ for some $p > 2n/(n-2)$. Trudinger [T, p. 271] proves this by a clever choice of the function z in (7.22). An alternate approach is given in [BN, Lemma 1.5], where they use a result of Brezis-Kato.

So far, we have a smooth solution $u \geq 0$ of

$$Lu = -\Delta u + cu = \beta f u^\alpha. \tag{7.23}$$

Just as in Proposition 5.3 (see equation (5.19)) we conclude that either $u \equiv 0$ or else $u > 0$ everywhere. Since $\int f u^N\, dx_g = 1$, $u > 0$. Finally, we eliminate the constant β . Since $c > 0$ and $f > 0$ then $\beta > 0$ (multiply (7.23) by u and integrate). Thus, letting $v = \beta^{1/(\alpha-1)} u$ we obtain a solution of (7.5). □

REMARK 7.1 If we delete the assumption that $c > 0$ in Corollary 7.6, then the above argument still proves the existence of a solution $u > 0$ of (7.23); however it is possible that $\beta > 0$, $\beta < 0$, or $\beta = 0$ —since β has the same sign as the lowest eigenvalue λ_1 of L . Scaling u we can reduce to $\beta = \pm 1$ or $\beta = 0$.

The last item in this section is a result of Kazdan-Warner that shows there are situations where equality holds in (7.12) and (7.5) does *not* have a solution, so J does not have a minimum in this case. Therefore, the difficulties in solving (7.5) using the calculus of variations are inherent in the problem and not just a defect of the method.

Theorem 7.7 [Kazdan-Warner [KW-6]] *Let* (S^n, g) *be the standard unit sphere in* R^{n+1} . *If* $u > 0$ *is a solution of* (7.5) *with* $c = n(n-2)/4$, *and* α *any exponent, then*

$$\int_{S^n} u^{\alpha+1} \nabla f \cdot \nabla \varphi \, dx_g + \tfrac{n-2}{2}\Big(\alpha - \tfrac{n+2}{n-2}\Big) \int_{S^n} u^{\alpha+1} f \varphi \, dx_g = 0 \tag{7.24}$$

for any first order spherical harmonic φ , *so* $-\Delta\varphi = n\varphi$. *In particular for* $\alpha = (n+2)/(n-2)$ *if* $f =$ *constant* $+\varphi > 0$, *there is no solution of* (7.5).

This theorem, which is still not very well understood, is obviously closely related to the obstruction (5.47) to solving $\Delta u = 2 - he^u$ on S^2 .

In view of the existence Theorem 7.5 and the non-existence Theorem 7.7, it is useful to have some information when strict inequality in (7.12) does hold. Aubin (see [Au-4]) did this by computing $J(\psi_t)$ where ψ_t is defined by equation (7.15). For $n > 4$ he found at x_0

$$J(\psi_t) = \Lambda_n f^{-2/N} \Big\{ 1 + \tfrac{2\rho^2}{n(n-4)} \big[\gamma c - S - \tfrac{(n-4)}{2} \tfrac{\Delta f}{f}\big] t^2 + o(t^2) \Big\}, \tag{7.25}$$

where $\gamma = 4(n-1)/(n-2)$, as in (7.4). We shall discuss this further in the next section. Notice that the scalar curvature appears in this equation even though it does not appear explicitly in our equation ((7.5)).

7.4 The Yamabe Problem, Geometric Part

To begin, we interpret the results of the previous section for the geometric problem (7.4). In this case, comparing (7.4) with (7.5) we have $S = \gamma c$ and $k = \gamma f$ is a constant. Thus, the term $[\cdots]t^2$ in (7.25) is zero. It is interesting to see how exactly the case of geometric interest causes the greatest trouble for (7.5). One is led to compute the next term in the expansion (7.25). Aubin carried this out and found that (for $S = \gamma c$ and $f \equiv$ constant) if $n \geq 6$, then

$$J(\psi_t) = \Lambda_n f(x_0)^{-2/N}\{1 - a_4|W|^2_{x_0}t^4 + o(t^4)\}, \tag{7.26}$$

where $|W|^2$ is the pointwise norm of the Weyl conformal curvature tensor and $a_4 > 0$ is a constant (actually, Aubin obtains (7.26) for a conformal metric). An immediate consequence is the following result.

Corollary 7.8 [Aubin] *If (M^n, g) is compact with $n \geq 6$ and $|W| \neq 0$ at some point, then there is a pointwise conformal metric with constant scalar curvature.*

Note that if $W = 0$ everywhere, then there are metrics arbitrarily near g with $|W| \neq 0$ somewhere, so, at least for $n \geq 6$, the Yamabe problem can be solved affirmatively for the generic metric.

Another corollary, due to Bérard-Bergery, shows that if we are given a metric g with positive scalar curvature, then we can find another metric g_1 with scalar curvature $S_1 \equiv 1$. This in some ways resembles Theorem 7.2 above, except that here the metrics g and g_1 are not necessarily pointwise conformal (if we knew that we can always solve the Yamabe problem (7.4), then we could find a pointwise conformal metric g_1). To prove this, we need a preliminary result. Let I_g be the functional

$$I_g(\varphi) = \frac{\int_M (\gamma|\nabla_g\varphi|^2 + S_g\varphi^2)\,dx_g}{(\int_M |\varphi|^N\,dx_g)^{2/N}}$$

associated with the equation (7.4) with the metric explicitly written, and let

$$\sigma(g) = \inf I_g, \tag{7.27}$$

just as in (7.9), except for some inessential alterations. In this notation, Theorem 7.5 tells us that we can solve the Yamabe problem (7.4) if

$$\sigma(g) < \gamma\Lambda_n = n(n-1)\omega_n^{2/n}. \tag{7.28}$$

Lemma 7.9 [Bérard-Bergery, see [Be-2]] *The number $\sigma(g)$ depends continuously on g.*

The proof is similar to the proof that the eigenvalues $\lambda_j(g)$ of the Laplacian Δ_g depend continuously on g.

Corollary 7.10 [Bérard-Bergery] *If (M^n, g), $n \geq 3$, has a metric with scalar curvature $S_g \geq 0 (\neq 0)$, then it has a metric g_1 with $S_{g_1} \equiv 1$.*

Proof. By Theorem 7.2, there is a metric g_- with $S_{g_-} < 0$ and hence $\sigma(g_-) < 0$. Also, $\sigma(g) > 0$. Let $g_t = tg_- + (1-t)g$. Then for some $0 < t \leq 1$ we know that $\sigma(g_t) > 0$ and $\sigma(g_t)$ also satisfies (7.28). Therefore, by Theorem 7.5 there is a metric g_1, pointwise conformal to g_t, with $S_{g_1} \equiv 1$. □

This last result enables us to give a reasonably complete answer to the question raised in Chapter 6.4 on prescribing scalar curvature— although the more delicate questions concerning solving (7.4) with a constant or more general function $k(x)$ are still unresolved.

In view of the topological restrictions of Section 7.2, we separate the compact manifolds into three groups:

(i) those M that have a metric g with $S_g \geq 0 (\not\equiv 0)$,

(ii) those M that have no metric with positive scalar curvature, but do have a metric with $S_g \equiv 0$,

(iii) the other manifolds, so for any metric g , the scalar curvature S_g is negative somewhere.

Theorem 7.11 KAZDAN-WARNER] *For manifolds M in class I, any function $S \in C^\infty$ is the scalar curvature of some metric. For M in class II, a function $S \in C^\infty$ is a scalar curvature if and only if either S is negative somewhere or $S \equiv 0$. For M in class III, a function $S \in C^\infty$ is a scalar curvature if and only if S is negative somewhere.*

Proof. Because we can use Theorem 7.2, Proposition 7.4, or Corollary 7.10 to obtain constant scalar curvature metrics, this result is an immediate consequence of Theorem 6.1. □

Chapter 8

Surfaces With Constant Mean Curvature

[*not yet revised*]

8.1 Introduction

For a two dimensional surface in $\mathbb{R}^3$, one has the concept of Gauss curvature, that depends on the metric, but not on the embedding in $\mathbb{R}^3$. In addition, one has the concept of *mean curvature* that does depend on the embedding. A right circular cylinder of radius R has zero Gauss curvature, since its metric is just that of the Euclidean plane, but its mean curvature $H = 1/R$.

Minimal surfaces are the simplest examples of surfaces with constant mean curvature: their mean curvature is zero. To be led to surfaces with constant non-zero mean curvature, one considers the problem of finding a compact surface of least surface area whose volume is a constant. Spheres are obvious examples: the sphere of radius R has mean curvature $1/R$.

This leads one to the natural question, are spheres the only compact surfaces with constant mean curvature? Since one may interpret mean curvature in terms of the surface tension of a soap film, this question can be more vividly stated: are spheres the only soap bubbles? We consider this in the first section.

The second section is devoted to a boundary value problem for surfaces of constant mean curvature.

8.2 Compact Surfaces

Let $M^2 \hookrightarrow \mathbb{R}^3$ be a compact surface with constant mean curvature. Then must M^2 be just the standard round sphere? (Of course we must exclude the zero mean curvature case, i.e. minimal surfaces, since for these, the Gauss curvature is never positive because *any* compact $M^2 \hookrightarrow \mathbb{R}^3$ has positive Gauss curvature K somewhere—for example, it has $K > 0$ at the point furthest from the origin).

The first results were by Liebmann in 1909, who proved that the only strictly convex possibility is the round sphere (see the references in [HTY]).

Next, in 1951, Heinz Hopf proved that if M^2 is the differentiable sphere, then the only possibilities are the standard round spheres. In fact, he only assumed that $S^2 \hookrightarrow \mathbb{R}^3$ was an immersion, not necessarily an embedding, so he allowed self-intersections. His proof uses complex analysis in a non-trivial way.

A.D. Alexandrov, in 1958, showed that the only hypersurface M^n *embedded* in $\mathbb{R}^{n+1}$ having constant mean curvature is the standard sphere. His beautiful proof is very geometric and elementary, using only the maximum principle for second order elliptic operators (see the exposition in [Sp]). The method has subsequently been useful for many problems. This proof belongs in these lectures but alas, there was no time. More recently, Reilly [Re] found an interesting different proof.

Since Alexandrov's work, it is natural to attempt to extend his results to cover the case where M^n is just *immersed* in $\mathbb{R}^{n+1}$, much as Hopf did in the special case of the sphere. But just the opposite occurred when recently, Hsiang, Teng, and Yu [HTY] found new examples of immersed compact hypersurfaces $M^n \hookrightarrow \mathbb{R}^{n+1}$, $n \geq 3$, with constant mean curvatures. In fact, they found non-round spheres $S^{2k-1} \hookrightarrow \mathbb{R}^{2k} k \geq 2$ with this property. Thus, Hopf's result is very special to the case of the sphere S^2, and the assumption that Alexandrov made that the hypersurfaces $M^n \hookrightarrow \mathbb{R}^{n+1}$ are embedded can not be omitted (except possibly in the still unresolved case $n = 2$).

Hsiang-Teng-Yu seek their examples as hypersurfaces of revolution. For $n \geq 3$, there are many possible definitions of a "hypersurface of revolution". The most flexible and useful definition is to view them as hypersurfaces that are invariant under some small subgroup of the orthogonal group. These new hypersurfaces $M^{2k-1} \hookrightarrow \mathbb{R}^{2k}$ are differentiable spheres that are invariant under the action of $O(k) \times O(k)$. The condition of constant mean curvature appears as an *ordinary* differential equation in this case.

There is much to be done before we really have a clear picture of all the possible constant mean curvature surfaces.

8.3 A Boundary Value Problem

Given a closed curve Γ in $\mathbb{R}^3$, we may seek a surface M having Γ as its boundary and having prescribed mean curvature H. The well-known *Plateau Problem*, that seeks a minimal surface with prescribed boundary is an example.

The above vague formulation does not at all specify the topological type of M. It has long been known that, for example, some curves are the boundaries of both orientable and non-orientable surfaces. Thus we will be more specific.

Let $\Omega \subset \mathbb{R}^2$ be the open unit disc, $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and $\gamma : \partial\Omega \to \mathbb{R}^3$ be a smooth curve. Given a smooth function $H(x, y)$ we seek a function $u : \overline{\Omega} \to \mathbb{R}^3$ having mean curvature H and agreeing with γ on $\partial\Omega$. The partial differential equations are (using the standard "cross product" of vectors in $\mathbb{R}^3$):

$$\Delta u = 2H \cdot u_x \times u_y \qquad \text{in } \Omega, \tag{8.1}$$

with Dirichlet boundary conditions

$$u = \gamma \qquad \text{on } \partial\Omega. \tag{8.2}$$

So far, most work has concentrated on the case $H \equiv$ constant > 0, so we will restrict our attention to that case (note that the sign of H is irrelevant, since it can be reversed simply by reversing the orientation of the surface). As an example, let $\gamma : (x, y) \mapsto (Rx, Ry, 0)$ be a circle of radius R. Then there are *two* solutions of (8.1)-(8.2), namely, the upper and lower spherical caps of a sphere of radius $a \geq R$ having mean curvature $H = 1/a$. If R and H are both large, it appears likely that there is *no solution.* In this special case where Γ is a circle, Heinz [Hz] proved that there is no solution if $HR > 1$. Of course if $HR < 1$ then we are in the above situation and have two solutions, while if $HR = 1$ the only obvious solution is the hemisphere.

In order to measure the "size" of the boundary curve, $\gamma(\partial\Omega)$, let us assume that it lies in a closed ball of radius R. S. Hildebrandt [Hi] proved, in 1970 that if $HR < 1$, then there is (at least) one solution to the Dirichlet Problem (8.1)–(8.2). We shall sketch his proof shortly.

Earlier, based on the spherical caps example above, Rellich had conjectured that for any curve $\Gamma = \gamma(\partial\Omega)$ there should exist at least *two* solutions, at least for H sufficiently small. The proof that this is true if $HR < 1$ (and $\gamma(\partial\Omega)$ is not a constant) was recently given by Brezis-Coron [BC] (independently, Struwe also proved that for some $c > 0$ if $HR < c$ then there are two solutions, but his method seems to give no information on c). If $\gamma(\partial\Omega)$ is a constant, then H. Wente showed that $u =$ constant is the *only* solution.

It would be interesting to know when these two solutions are the *only* solutions. Even in the case where $\gamma(\partial\Omega)$ is a round circle we do not know if there are solutions other than the two spherical caps. There are many natural questions in this area that call one's attention.

Hildebrandt's proof of the existence of at least one solution of the Dirichlet Problem (8.1)-(8.2), assuming $H > 0$ is a constant and $HR < 1$, uses the calculus of variations. He minimizes the functional

$$E(v) = \int_\Omega |\nabla v|^2\, dx + \tfrac{4}{3} H \int_\Omega v \cdot (v_x \times v_y)\, dx \tag{8.3}$$

in the set of vector-valued functions $v \in H^{2,1}(\Omega)$ such that $v = \gamma$ on $\partial\Omega$ and $\|v\|_{L^\infty} \le R'$ for some $R' > R$ with $HR' < 1$. It is obvious that

$$E(v) \ge \left(1 - \tfrac{2}{3} H \|v\|_{L^\infty}\right) \int_\Omega |\nabla v|^2\, dx \ge \tfrac{1}{3} \int_\Omega |\nabla v|^2\, dx, \tag{8.4}$$

that shows that E is bounded below. Let $\sigma = \inf E$ and say $E(v_j) \downarrow \sigma$. From (8.4) and the fact that $\|v_j\|_{L^\infty} \le R'$ the sequence lies in a bounded set in $H^{2,1}(\Omega)$, so there is a subsequence that converges weakly in $H^{2,1}(\Omega)$. The only tricky part of the remainder of the proof is how one treats the last integral in (8.3) for the sequence v_j (see [Hi] or [BC] for details).

To prove the existence of at least *two* solutions, Brezis-Coron use the above solution, that we call w, as their first solution and seek a second solutions u as $u = w + v$. Substituting this into (8.1) we find that v must satisfy

$$Lv := -\Delta v + 2H(w_x \times v_y + v_x \times w_y) = -2H(v_x \times v_y) \tag{8.5}$$

in Ω with

$$v = 0 \quad \text{on } \partial\Omega. \tag{8.6}$$

We seek a variational problem for v —but want to avoid getting the obvious solution $v \equiv 0$ of (8.5). The linear differential operator L in (8.5) is self-adjoint and is the Euler-Lagrange operator for the functional

$$J(v) = \int_\Omega [|\nabla v|^2 + 4Hw \cdot (v_x \times v_y)]\, dx. \tag{8.7}$$

(Note $J(v) = \langle Lv, v \rangle$ after an integration by parts.) On the other hand, the right side of (8.5) is the Euler-Lagrange operator for the functional

$$Q(v) = \int_\Omega v \cdot (v_x \times v_y)\, dx. \tag{8.8}$$

($Q(v)$ is often described as a type of "volume", perhaps because it is cubic in v, but I am not sure.)

Thus, it is reasonable to minimize the quotient

$$P(v) = \frac{J(v)}{Q(v)^{2/3}} \tag{8.9}$$

or, equivalently, to minimize J on the set where $Q = 1$. The Euler-Lagrange equation for this is exactly (8.5), except that one may have to replace v by constant v to eliminate the Lagrange multiplier.

As usual, the first step is to show that J is bounded below. One proves that for some constant $c > 0$

$$J(v) \geq c\|v\|^2_{H^{2,1}(\Omega)} \tag{8.10}$$

for all v in $H^{2,1}(\Omega)$ (here $H^{2,1}(\Omega)$ is a slightly modified version of $H^{2,1}(\Omega)$ to incorporate the boundary condition $v = 0$ on $\partial\Omega$; it is defined as the completion of $C_0^\infty(\Omega)$ in the $H^{2,1}(\Omega)$ norm).

Inequality (8.9) proves that a minimizing sequence is bounded in the space $H^{2,1}(\Omega)$ and hence has a weakly convergent subsequence. The problem comes from the fact that functional $Q(v_j)$ is not continuous under this convergence, just as we saw in the Yamabe problem for the functional (7.6). In fact, the analogy is much closer than one might expect, and the proof here was strongly influenced by that of Theorem 7.5.

Let

$$S = \inf \frac{\int |\nabla\varphi|^2\, dx}{Q(\varphi)^{2/3}} \tag{8.11}$$

for all $\varphi \in H^{2,1} \cap L^\infty$, that is similar to (7.10) where we used the best constant in a Sobolev inequality. One can view (8.10) as an *isoperimetric inequality* and show that $S = (32\pi)^{1/3}$. There is no map $\varphi : \Omega \to \mathbb{R}^3$ giving this constant (in this case the integrals in (8.10) are over $\mathbb{R}^2$, not Ω). In fact there is a family of such maps

$$\Psi_t(x, y) = \frac{(x, y, t)}{x^2 + y^2 + t^2}, \qquad t \neq 0. \tag{8.12}$$

These play the same role as the functions Ψ_t in (7.15). Parallel to proving that $\sigma \leq \Lambda_n$ (i.e. (7.12) with $f = 1$), here one uses the functions (8.11) to show that

$$\sigma < S, \tag{8.13}$$

where $\sigma = \inf J(v)$ for v satisfying $Q(v) = 1$. Armed with this inequality, one can prove that there is a map $v : \Omega \to \mathbb{R}^3$ minimizing (8.8) with $Q(v) = 1$. Of course, the condition $Q(v) = 1$ shows that $v \not\equiv 0$ and hence that $u = w + v$ is a distinct second solution of the problem. For more details, we refer to the paper [BC].

Chapter 9

Ricci Curvature

[*not yet revised*]

9.1 Introduction

Since we now have some reasonable understanding of the scalar curvature, it is time to consider seriously the Ricci curvature. The local problem of solving

$$\mathrm{Ric}(g) = R_{ij} \tag{9.1}$$

was discussed in Chapter 6.5. Here we will consider a few global questions. First, we recall that (M^n, g) has *constant Ricci curvature* if

$$\mathrm{Ric}(g) = \lambda g \tag{9.2}$$

for some constant λ. Metrics having constant Ricci curvature are customarily called *Einstein metrics*. Taking the trace of (9.2) we obtain $\lambda = S/n$ where S is the scalar curvature. Thus (9.2) the same as

$$\mathrm{Ric}(g) = \tfrac{1}{n} S g. \tag{9.3}$$

If $n = 2$, then (9.3) is always satisfied—except that S may not be a constant—while for $n \geq 3$, if (9.3) holds then the second Bianchi identity (A.31) shows that $S \equiv \text{const.}$ anyway. From now on, we assume $n \geq 3$ and M is compact (and connected, of course).

Topological obstructions. Bochner's result (see Theorem 3.3 shows there are topological obstructions to positive Ricci curvature. After Aubin's earlier work showing there are no topological obstructions to negative scalar curvature and Gao-Yau [Gao, Gao-Yau] for Ricci curvature in the three dimensional case, Lohkamp [Lo] proved that every smooth compact manifold of dimension at least three has a smooth metric with negative Ricci curvature, so there are no topological obstructions to negative Ricci curvature. Note that there are topological obstructions to negative sectional curvature.

One can also ask if there are topological obstructions to manifolds having Einstein metrics. We first discuss the case $n = 3$. In this case one can associate with every *2*-dimensional plane a unique normal direction. Thus, in an orthonormal basis e_1, e_2, e_3 one can write the Ricci curvature, viewed as a quadratic form, in terms of the sectional curvatures as $\mathrm{Ric}(e_1) = \mathrm{sect}\,(e_1, e_2) + \mathrm{sect}\,(e_1, e_3)$, with similar formulas for the other directions, e_2, e_3. Solving the resulting three equations gives an explicit formula for the sectional curvature in terms of the Ricci curvature. In tensor notation this formula is

$$R_{ijk\ell} = g_{ik}R_{j\ell} - g_{i\ell}R_{jk} - g_{jk}R_{i\ell} + g_{j\ell}R_{ik} - \tfrac{1}{2} S (g_{ik}g_{j\ell} - g_{i\ell}g_{jk}), \tag{9.4}$$

This formula shows that an Einstein metric in dimension 3 must have constant sectional curvature. But then its universal cover is either $\mathbb{R}^3$ or the sphere S^3. Consequently, $S^2 \times S^1$, whose universal cover is $S^2 \times \mathbb{R}$, can not have an Einstein metric. It is unknown if, for example, S^3 and T^3 have negative Einstein metrics (negative means negative scalar curvature).

For $n = 4$, Thorpe (see [Be-2]) found some topological obstructions to Einstein metrics. This was later rediscovered and improved by Hitchin. Subsequently Polombo showed that these are obstructions to the Ricci curvature being too pinched, not just Einstein metrics. There are no known topological obstructions if $n \geq 5$, but one expects there are some that have not yet been found.

By now, it should be evident that there are more questions than answers, even to the most obvious questions concerning Ricci curvature.

Uniqueness. Say there are two metrics g_1 and g_2 with

$$\operatorname{Ric}(g_1) = \operatorname{Ric}(g_2). \tag{9.5}$$

Since $\operatorname{Ric}(cg_1) = \operatorname{Ric}(g_1)$ for any constant $c > 0$, the most we can expect to conclude is that $g_1 = \text{const.} g_2$. Hamilton proved that if g_0 is the standard metric on S^n and if $\operatorname{Ric}(g) = \operatorname{Ric}(g_0)$ on S^n, then $g = cg_0$ for some constant $c > 0$, so uniqueness does hold in this situation. This was subsequently extended by DeTurck-Koiso [DKo]. One consequence in their work is that certain positive definite symmetric tensors can *not* be the Ricci tensors of a Riemannian metric; this is a *non-existence* result for the equation $\operatorname{Ric}(g) = R_{ij}$. The proofs use Hamilton's observation that if $\operatorname{Ric}(g)$ is positive definite, and hence may also be viewed as a metric itself, then the second Bianchi identity (A.30) states that the identity map

$$(M, g) \xrightarrow{id.} (M, \operatorname{Ric}(g))$$

is a *harmonic map* (see [EL-1] for the definition). For more information see the book [Be-2].

Regularity. If $\operatorname{Ric}(g)$ is smooth, must g be smooth? A simple example shows that this may depend on the coordinates used. Let g_0 be the standard flat metric on the torus T^n (so $g_0 = \sum_i (dx^i)^2$ in the usual local coordinates and is real analytic). If $\varphi : T^n \to T^n$ is a diffeomorphism of class C^k but not of class C^{k+1}, then $\operatorname{Ric}(\varphi^*(g_0)) \equiv 0$, that certainly is smooth, while the metric $g = \varphi^*(g_0) \in C^{k-1}$ but $g \in C^k$. This example also shows that Einstein metrics need not be smooth, since here g is also an Einstein metric.

This issue was clarified by DeTurck-Kazdan [DK], who showed that it is important to pick "good" local coordinates. If $g = \sum_{ij} g_{ij}(x)\, dx^i dx^j$, then we say that $g \in C^k$ (or C^∞, or $C^\omega =$ real analytic) in these coordinates if the functions $g_{ij}(x) \in C^k$ (or C^∞ etc.) in these coordinates. The main point is that *harmonic coordinates* are optimal for regularity questions. Harmonic coordinates are, by definition, where the coordinate functions $x^1, \ldots x^n$ are harmonic functions, $\Delta x^k = 0$. Isothermal coordinates if $n = 2$ are an example (see Chapter 6.2).

Proposition 9.1 (DK) *If in some coordinate chart $g \in C^{k,\alpha}$, $k \geq 1$ (or C^ω), and a tensor $T \in C^{k,\alpha}$ (or C^ω) then in harmonic coordinates we also have $T \in C^{k,\alpha}$ (or C^ω). In particular, $g \in C^{k,\alpha}$ (or C^ω) in harmonic coordinates.*

The proof follows from the observation that if a function $u \in C^2$ satisfies $\Delta u = 0$, then in local coordinates (1.6)

$$\sum_{i,j} \frac{\partial}{\partial x^i}(g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x^j}) = 0.$$

Thus if $g \in C^{k,\alpha}$, then u is the solution of a linear elliptic equation whose coefficients are in $C^{k-1,\alpha}$ so, by elliptic regularity (Theorem 2.3), $u \in C^{k+1,\alpha}$. Thus the isometric map φ from the given coordinates $(x^1, \ldots, x^n)$ to harmonic coordinates $(u^1, \ldots, u^n)$ is of class $C^{k+1,\alpha}$. Transforming a tensor T of rank at least one involves only the first derivatives of the map φ, so $T \in C^{k,\alpha}$ (or C^ω) in harmonic coordinates too.

Note that geodesic normal coordinates are *not* optimal; Example (2.3) in [DK] shows that one can *lose* two derivatives in these coordinates.

As we saw in Chapter 6.5 when we computed the symbol of the differential operator $\operatorname{Ric}(g)$, this operator is not elliptic because of its invariance under the group of diffeomorphisms. However if one restricts to harmonic coordinates, then $\operatorname{Ric}(g)$ is elliptic.

Proposition 9.2 [DK] *In harmonic coordinates,*

$$\operatorname{Ric}(g)_{ij} = -\tfrac{1}{2} g^{st} \frac{\partial^2 g_{ij}}{\partial x^s \partial x^t} + Q(g, \partial g), \tag{9.6}$$

where Q is a real analytic function of its variables; in fact, it is a polynomial except, because of the presence of g^{-1}, it involves $\det g$ in the denominator. From this formula, in harmonic coordinates $\operatorname{Ric}(g)$ is an elliptic operator.

Proof. Using the standard formula for the Laplacian (1.6) we find that $\Delta x^i = 0$ implies

$$g^{st}\Big(\frac{\partial g_{is}}{\partial x^t} - \tfrac{1}{2}\frac{\partial g_{st}}{\partial x^i}\Big) = 0.$$

Taking the partial derivative of this with respect to x^j and adding it to a similar formula with the roles of i and j interchanged we find that

$$g^{st}\Big(\frac{\partial^2 g_{is}}{\partial x^t \partial x^j} + \frac{\partial^2 g_{js}}{\partial x^t \partial x^i} - \frac{\partial^2 g_{st}}{\partial x^i \partial x^j}\Big) = -\frac{\partial g^{st}}{\partial x^j}\frac{\partial g_{si}}{\partial x^t} + \cdots ,$$

where the remaining terms on the right are similar to the first term in the right. In particular, the right hand side depends only on g and its first derivatives. We now use this in the explicit formula for $\operatorname{Ric}(g)$ (A.28) to obtain (9.6). □

As an immediate application, by using elliptic regularity (Theorem 4.1) we obtain the following.

Corollary 9.3 [DK]
a*). If in harmonic coordinates $g \in C^2$ is a Riemannian metric with $\operatorname{Ric}(g) \in C^{k,\alpha}$ for some $k \geq 0$ (or $\operatorname{Ric}(g) \in C^\omega$), then in these coordinates $g \in C^{k+2,\alpha}$ (or C^ω).*
b*). In harmonic coordinates Einstein metrics are real analytic.*

REMARK 9.1 In part a) one can avoid the use of harmonic coordinates if instead one assumes $\operatorname{Ric}(g)$ is invertible. For this, say $\operatorname{Ric}(g) = R$ and use DeTurck's device of introducing the equivalent elliptic operator $\operatorname{Ric}(g) + \delta^*\big(R^{-1}\operatorname{Bian}(g,R)\big)$, where $2\operatorname{Bian}(g,R)$ is the expression in the Bianchi identity (6.29) and δ^*. the symmetric covariant derivative of a $1-$ form is defined by $(\delta^* w)_{ij} = \frac{1}{2}(w_{i;j} + w_{j;i})$. Since $\operatorname{Bian}(g,R) = 0$, this operator is $\operatorname{Ric}(g)$, yet it is elliptic.

9.2 Positive Einstein Metrics on M^3

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A basic question is if a given manifold M^n has an Einstein metric—or better yet, a metric of constant sectional curvature. The formula (9.4) expressing the sectional curvature in terms of the Ricci curvature in dimension three shows that in this special dimension the constant sectional curvature metrics are just the Einstein metrics. This question is important because if M^n admits an Einstein or constant sectional curvature metric, then one can use the metric to help read off properties of M. For example, as we mentioned previously if M^n is simply connected and admits a metric with constant positive sectional curvature, then M^n must be the sphere S^n.

Some progress has been made if $n = 3$, and on Kähler manifolds. In this section we discuss the recent work of Hamilton [H-2] on positive Einstein metrics on three dimensional manifolds, while the next section takes up the Kähler-Einstein case.

Theorem 9.4 (H-2) *Let (M^3, g_0) be compact with positive Ricci curvature. Then there is a family of metrics g_t, $0 \le t \le \infty$, with positive Ricci curvature and with g_∞ an Einstein metric.*

In view of a result of Aubin [Au-3], it is actually sufficient to assume that $\mathrm{Ric}(g_0) \ge 0$, with $\mathrm{Ric}(g_0) > 0$ at one point, because then Aubin shows that one can deform g_0 to a metric with everywhere strictly positive Ricci curvature. On the other hand, the manifold $S^2 \times S^1$, whose standard metric has $\mathrm{Ric}(g_0) \ge 0$ but does not admit an Einstein metric (see Section 9.1 above), shows that one needs $\mathrm{Ric}(g_0)$ strictly positive somewhere.

Hamilton thus wants to solve (9.3) with $S > 0$. It is reasonable to use the heat equation and seek the metrics g_t by solving the initial value problem

$$\frac{\partial g_t}{\partial t} = 2[\tfrac{1}{3}S(g_t)g_t - \mathrm{Ric}(g_t)] \tag{9.7}$$

with

$$g_t|_{t=0} = g_0 \tag{9.8}$$

$S(g)$ is, as usual, the scalar curvature and the factor 2 is for convenience later. Instead of using the heat equation, one could try the continuity method to solve (9.2), but this has not yet been done.

Using (9.7) one can derive an equation for $\partial S/\partial t$. This is a "backward" heat equation, and we mentioned earlier (Chapter 3.5) that the initial value problem is not always solvable for such equations. Instead, since $S(g_\infty)$ will be a constant, Hamilton replaces $S(g)$ by its average

$$r(g) = \frac{1}{\mathrm{Vol}(g)} \int_M S(g)\, dx_g$$

and solves

$$\frac{\partial g}{\partial t} = 2[\tfrac{1}{3}r(g)g - \mathrm{Ric}(g)] \tag{9.9}$$

(we will sometimes write g instead of g_t) with the initial condition (9.8). It is easy to check that for these metrics $d\,\mathrm{Vol}(g_t)/dt = 0$ so, scaling g_0 if necessary, we have the normalization $\mathrm{Vol}(g_t) = \mathrm{Vol}(g_0) = 1$.

For computations it is often more convenient to treat the simpler unnormalized equation

$$\frac{\partial g}{\partial t} = -2\ \mathrm{Ric}(g). \tag{9.10}$$

To go from (9.10) to (9.9) one makes the change of scale $g^* = \psi g$, choosing $\psi(t) > 0$ to satisfy the normalization condition $\mathrm{Vol}(g^*) = 1$ and use a new time variable t^* defined by $t^* = \int \psi(t)dt$. Then a calculation shows that g^* satisfies

$$\frac{\partial g^*}{\partial t^*} = 2[\tfrac{1}{3}r(g^*)g^* - \mathrm{Ric}(g^*)], \tag{9.11}$$

as desired.

We follow the same steps as in the model problem in Chapter 5 to solve (9.9), (9.8), except that here everything is considerably more complicated.

Step 1 is to prove the short time existence of a solution to (9.10) with initial conditions (9.8). Now equation (9.10) is almost but not quite, parabolic. The difficulty is with degeneracies caused by the group of diffeomorphisms, just as we saw in Chapter 6.5 for the equation $\mathrm{Ric}(g) = R_{ij}$. Hamilton used the Nash-Moser implicit function theorem for this step. Subsequently, DeTurck [D-4] simplified this and showed how one can use standard parabolic theory. He lets T_{ij} be any invertible symmetric tensor, such as $T = g_0$, and solves the initial value problem

$$\frac{\partial g}{\partial t} = -2[\mathrm{Ric}(g) - \sigma^*(T^{-1}\mathrm{Bian}\,(g,T))], \tag{9.12}$$

where $2\mathrm{Bian}\,(g,R)$ is the expression in the Bianchi identity (6.29) and, for a *1*-form w, we define $\sigma^* w$ by $(\sigma^* w)_{ij} = \frac{1}{2}(w_{i;j} + w_{j;i})$, that is, $\sigma^* w$ is the symmetric covariant derivative. The virtue of (9.12) is that it is a parabolic equation, so by Theorem 4.6 there is a solution g_t of (9.12) satisfying (9.8). To go from (9.12) to (9.10) we use a diffeomorphism φ_t defined by

$$\frac{d\varphi_t}{dt} = v(\varphi_t(x), t), \qquad \varphi_0 = id,$$

where v is the vector field dual to the *1*-form $-T^{-1}\mathrm{Bian}\,(g_t, T)$. A computation then shows that if g_t satisfies (9.12) then the metric $\varphi_t^*(g_t)$ satisfies (9.10) and (9.8) for at least some short time interval $0 \le t < \epsilon$; this consequently gives a solution of (9.11) in a small time interval.

To get more information, one needs formulas for the evolution of $\mathrm{Ric}(g_t)$ and $S(g_t)$, where g_t is a solution of (9.10). In dimension three they are genuine heat equations;

$$\frac{\partial\,\mathrm{Ric}(g_t)}{\partial t} = \Delta\,\mathrm{Ric}(g_t) - Q, \tag{9.13}$$

where Q is a polynomial in $g_t, \mathrm{Ric}(g_t)$, and $S(g_t)$ (but does not contain any additional derivatives of these) and

$$\frac{\partial S(g_t)}{\partial t} = \Delta S(g_t) + 2|\mathrm{Ric}(g_t)|^2. \tag{9.14}$$

From (9.13) and a version of the maximum principle for symmetric tensors, Hamilton shows that if $\mathrm{Ric}(g_0)$ is positive, then so is $\mathrm{Ric}(g_t)$ for $t > 0$. This guarantees that all of our metrics g_t, and also the corresponding metrics $g_t^* = \psi g_t$, will have positive Ricci curvature.

Step 2 is to show that this solution g^* of (9.11) exists of all $0 \le t^* < \infty$. To do this, say a solution exists on some maximal interval $0 \le t^* < T^*$. By difficult but elementary arguments using the maximum principle one obtains appropriate *a priori* estimates on the solution g_t of (9.10) and on $\mathrm{Ric}(g_t)$ and their derivatives and can conclude that $T^{*=\infty}$. (The arguments in [H-2] can be simplified somewhat — for instance one can prove Lemma

16.7 in [H-2] without the sphere theorem by proving that $r(g^*)$ is an increasing function of t^* and observing that $r(g^*) \le \max_M S(g^*)$. However, to be candid, I have not yet checked all the details.)

Step 3 is to prove that the g^* converge to an Einstein metric as $t^* \to \infty$. This involves two types of estimates. One of them proves that pointwise the three eigenvalues $\lambda_j(x,t^*), j = 1,2,3$, of $\mathrm{Ric}(g^*)$ converge to some common value, $\lambda_j(x,t^*) \to \lambda(x), j = 1,2,3$, and that $S(g^*) \to 3\lambda(x)$ as $t^* \to \infty$. The second estimates are on the derivative of the scalar curvature $S(g^*)$ to show that $\lambda(x) \equiv$ constant. The inequalities used to prove these assertions are rather complicated, although doubtlessly will be simplified by subsequent work. After this, it is relatively straightforward to prove that the metrics g^* converge to an Einstein metric.

The main obstacle to extend this to dimensions higher than three is that we can no longer use (9.4) to replace the sectional curvature by the Ricci curvature. In addition, some higher dimensional spheres are known to have several *different* positive scalar curvature Einstein metrics. This may cause complications in proving convergence of metrics to Einstein metrics because there are now several possible targets. We should also make the obvious remark that if one can prove that every compact simply connected *3*-manifold M has a metric with positive Ricci curvature, then by Hamilton's Theorem 9.4 M is S^3. This is, of course, the Poincaré conjecture. It is not at all clear how one can fill the gap by proving the existence of a metric with positive Ricci curvature.

9.3 Kähler-Einstein Metrics

a) Some background on Kähler geometry

Let M^{2m} be a manifold of real dimension $2m$ and let $(x^1,\dots,x^m,y^1,\dots,y^m)$ be local coordinates. Write $z^k = x^k + iy^k, k = 1,\dots,m$ so $z = (z^1,\dots,z^m)$ are complex local coordinates. M^{2m} is a *complex manifold* if there is an atlas so that the change of coordinates is by analytic functions. Then m is the *complex dimension*. If u is a function we can write

$$du = \sum_k \left(\frac{\partial u}{\partial z^k}\, dz^k + \frac{\partial u}{\partial \bar z^k}\, d\bar z^k\right), \tag{9.15}$$

where, by definition,

$$\frac{\partial}{\partial z^k} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar z^k} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right).$$

We also define

$$\partial u = \sum_k \frac{\partial u}{\partial z^k}\, dz^k \qquad \text{and} \qquad \bar\partial u = \sum_k \frac{\partial u}{\partial \bar z^k}\, d\bar z^k,$$

so $d = \partial + \bar\partial$. A differential *2*-form of the type $dz^k \wedge dz^\ell$ is called of *type* $(2,0)$, while $dz^k \wedge d\bar z^\ell$ of type $(1,1)$ and $d\bar z^k \wedge d\bar z^\ell$ of type $(0,2)$. Now

$$0 = d^2 u = \partial^2 u + (\partial\bar\partial + \bar\partial\partial)u + \bar\partial^2 u. \tag{9.16}$$

Since $\partial^2 u$ is of type $(2,0)$, $(\partial\bar\partial + \bar\partial\partial)u$ of $(1,1)$ and $\bar\partial^2 u$ of type $(0,2)$, we conclude that $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$.

If g is a Hermitian metric on M, then

$$g = ds^2 = 2\sum g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta,$$

where $g_{\alpha\beta} = g_{\overline{\alpha}\overline{\beta}} = 0$ and $g_{\alpha\overline{\beta}} = g_{\overline{\beta}\alpha} = \overline{g_{\beta\overline{\alpha}}}$ for $\alpha, \beta, \overline{\alpha}, \overline{\beta}$ running from 1 to m.

The *Kähler* (or *fundamental*) form associated with g is the $(1,1)$ form

$$\gamma_g = \tfrac{i}{2\pi} \sum g_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta. \tag{9.17}$$

This is a real form since $\overline{\gamma_g} = \gamma_g$. It is *positive* since $g_{\alpha\overline{\beta}}$ is positive definite. The manifold (M,g) is said to be a *Kähler manifold* if γ_g is closed: $d_{\gamma_g} = 0$. There are many equivalent ways to write this Kähler condition. We simply refer to standard books (as well as the seminar [SP]) for generalities and examples and just list the facts we actually need.

FACT 1 The *Kähler Laplacian* on a function u is

$$\Delta_K u = \sum g^{\alpha\overline{\beta}} \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^\beta}. \tag{9.18}$$

One can show that $\Delta u = 2\Delta_K u$, where Δ is the real Laplacian we have been using.

FACT 2 The *Ricci curvature* is given by the formula

$$R_{\alpha\overline{\beta}} = -\frac{\partial^2 \log(\det g_{\alpha\overline{\beta}})}{\partial z^\alpha \partial \overline{z}^\beta}. \tag{9.19}$$

(The simplicity of this formula—compared to the much more complicated one for general Riemannian manifolds—is the reason one can often prove many results in Kähler geometry). For convenience we will often use the notation u'' for the complex hessian of a function

$$u'' = \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^\beta}.$$

Then (9.19) reads

$$\mathrm{Ric}(g) = -(\log\, \det g)''.$$

Just as g and γ_g are related, we define the *Ricci form* to be the $(1,1)$ form

$$\rho_g = \tfrac{i}{2\pi} \sum R_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta, \tag{9.20}$$

so ρ_g is a real form. In view of (9.19) we have

$$\rho_g = -\tfrac{i}{2\pi} \partial\overline{\partial} \log(\det g). \tag{9.21}$$

FACT 3 Let ω be a closed $(1,1)$ form on a compact Kähler manifold. Then ω is exact (or cohomologous to zero) if and only if there is a function h so that $\omega = \partial\overline{\partial} h$. One proves this using Hodge theory; it is obvious that if $\omega = \partial\overline{\partial} h$ then $\omega = d\overline{\partial} h$ so ω is exact.

FACT 4 The cohomology class of the Ricci form (on a compact Kähler manifold) is independent of the metric, since if g and g_1 are two Kähler metrics, then from (9.21)

$$\begin{aligned} \rho - \rho_1 &= \tfrac{i}{2\pi} \partial\overline{\partial}\, \log[(\det g)/(\det g_1)] \\ &= \tfrac{i}{2\pi} \partial\overline{\partial}\, \log f \end{aligned} \tag{9.22}$$

where f is the real valued function $(\det g)/(\det g_1)$. This cohomology class of closed $(1,1)$ forms is called the *first Chern class*, and written $c_1(M)$ — or sometimes just c_1. In complex dimension one $=$ real dimension 2, this is just the Gauss-Bonnet theorem. We say that $c_1(M)$ is *positive* if there is a positive $(1,1)$ form in $c_1(M)$; the definition that $c_1(M)$ is *negative* is obvious.

FACT 5 The *volume form* dv_g of a Kähler manifold (M, g) is

$$dv_g = a_m(\det g)\, dz^1 \wedge \ldots \wedge dz^m \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^m, \tag{9.23}$$

and also

$$dv_g = b_m(\gamma_g)^m, \tag{9.24}$$

where a_m and b_m are constants depending only on the dimension.

From now on M will be a compact connected Kähler manifold of complex dimension m.

b) Calabi's Problem: the formulation

We know that the Ricci form ρ_g represents $c_1(M)$. Calabi asked if the converse is true:

CALABI'S PROBLEM: Let ω be a closed real $(1, 1)$ form that represents $c_1(M)$. Is there a Kähler metric g whose Ricci form is ω, so $\rho_g = \omega$?

We formulate this as a partial differential equation. Since M is Kähler, there is some Kähler metric g_0. We seek g cohomologous to g_0, that is, γ_g is cohomologous to γ_{g_0}. Then by Fact 3 there is a real function φ so that

$$\gamma_g - \gamma_{g_0} = \tfrac{i}{2\pi}\partial\overline{\partial}\varphi, \tag{9.25}$$

or equivalently, in the notation introduced after (9.19),

$$g - g_0 = \varphi''. \tag{9.26}$$

Also ρ_{g_0} and ω both represent $c_1(M)$, so by Fact 3

$$\omega - \rho_{g_0} = -\tfrac{i}{2\pi}\partial\overline{\partial} f \tag{9.27}$$

for some real function f. If $\omega = \rho_g$, then by (9.22)

$$\omega - \rho_{g_0} = \rho_g - \rho_{g_0} = -\tfrac{i}{2\pi}\partial\overline{\partial}\log(\det g/\det g_0).$$

Combining this with (9.26) and (9.27) we conclude that

$$\partial\overline{\partial} f = \partial\overline{\partial}\log[\det g_0 + \varphi")/\det g_0].$$

Taking the trace of both sides, from (9.18) one finds $\Delta(f - \log[\quad]) = 0$ so $f - \log[\quad] =$ constant. Incorporating this constant into f we find that φ must satisfy the *Monge-Ampère equation*

$$\det(g_0 + \varphi'') = (\det g_0)e^f. \tag{9.28}$$

But since γ_g and γ_{g_0} are cohomologous, then by Fact 5 $\mathrm{Vol}(M, g_0) = \mathrm{Vol}(M, g)$, that is

$$\int_M e^f\, dv_{g_0} = \mathrm{Vol}(M, g_0). \tag{9.29}$$

This can always be arranged by adding a constant to f.

In the case of complex dimension $m = 1$ this equation is the elementary linear equation $1 + \Delta_K\varphi = e^f$, that is

$$\Delta\varphi = 2e^f - 2,$$

whose solvability is evident.

The solution φ of (9.28), if one exists, is unique up to an additive constant. This is obvious from part *b)* of the Comparison Theorem 4.4. The first complete existence proof was by S.T. Yau [Y].

Theorem 9.5 *(Calabi's Problem). Let* (M, g_0) *be a compact Kähler manifold and* ω *a closed real* $(1,1)$ *form that represents* $c_1(M)$ *. Then there is a unique Kähler metric* g *cohomologous to* g_0 *(i.e.* $\gamma_g \cong \gamma_{g_0}$ *) whose Ricci form is* ω *.*

One consequence is that there exist Riemannian manifolds whose Ricci curvature is everywhere zero, but whose sectional curvature is not everywhere zero. While this is not surprising, there were no examples prior to this result. To give an example, consider a *K-3* surface. It is a compact Kähler manifold with $c_1 = 0$ and $\hat{A} \neq 0$ so by the above, there is a Kähler metric with zero Ricci curvature. This metric can not have zero sectional curvature since if it did, all the Pontryagin classes (these are polynomials in the curvature) would be zero. But then we have a contradiction $\hat{A} \neq 0$ because $\hat{A}$ is expressed in terms of a Pontryagin class.

c) Kähler-Einstein metrics

Given a compact Kähler manifold, M , we seek a Kähler-Einstein metric, so, for some constant λ

$$R_{\alpha\overline{\beta}} = \lambda g_{\alpha\overline{\beta}}, \qquad \text{or equivalently,} \qquad \rho_g = \lambda \gamma_g. \tag{9.30}$$

Since ρ_g represents $c_1(M)$ and γ_g is positive, this means that $\lambda c_1(M)$ must be positive (unless $\gamma = 0$).

If $\lambda = 0$ then $c_1(M) = 0$ and we know from Calabi's problem that there is a Kähler-Einstein metric, the one with Ricci curvature zero. Thus we assume $\lambda \neq 0$.

QUESTION: If $c_1(M)$ is positive (or negative), does M have a Kähler-Einstein metric?

We write this as a partial differential equation. There is some Kähler metric; its Ricci form ρ_0 represents $c_1(M)$ and hence must be positive or negative so we can define a new metric g_1 and corresponding Kähler form by $\gamma_{g_1} = \rho_0/\lambda$. Then $\lambda\gamma_{g_1}$ represents $c_1(M)$, as does ρ_{g_1} . Thus there is a real function f so that

$$\rho_{g_1} - \lambda\gamma_{g_1} = i\,\partial\overline{\partial}\, f. \tag{9.31}$$

If the desired Kähler-Einstein metric g exists, then $\lambda\gamma_g = \rho_g$ so $\lambda\gamma_g$ also represents $c_1(M)$. This means there is a real function φ so that

$$\gamma_g - \gamma_{g_1} = i\,\partial\overline{\partial}\varphi, \quad \text{that is,} \quad g - g_1 = \varphi''. \tag{9.32}$$

Thus

$$\rho_g - \rho_{g_1} = \lambda\gamma_g - (\lambda_{\gamma_{g_1}} + i\,\partial\overline{\partial}\, f) = i\,\partial\overline{\partial}\,(\lambda\varphi - f).$$

But also

$$\rho_g - \rho_{g_1} = -\partial\overline{\partial}\,\log(\det g/\det g_1).$$

Combining the last two equations we see that φ must satisfy the Monge-Ampère equation

$$\det(g_1 + \varphi'') = (\det g_1)e^{f-\lambda\varphi}. \tag{9.33}$$

In the special case of complex dimension one, this reads (recall $\Delta = 2\Delta_K$)

$$1 + \tfrac{1}{2}\Delta\varphi = e^{f-\lambda\varphi}$$

In Chapter 5.7 we already saw that the case $\lambda < 0$ is much easier than the case $\lambda > 0$, where we found the obstruction (5.47) to solving this equation on S^2 . This situation persists in higher dimensions T . Aubin proved that if $\lambda < 0$ then equation (9.32) has exactly one solution (the uniqueness is an immediate consequence of part *b)* of Theorem

4.4, while there are several Kähler manifolds with $c_1 > 0$ that do not admit Kähler-Einstein metrics. Futaki [Fu] found an extension of (5.47) to the higher dimensional case (9.32) with $\lambda > 0$, and gave valuable new examples and insight into the solvability of (9.33) and the existence of positive Kähler-Einstein metrics. Let us formally state the existence assertion.

Theorem 9.6 (Au-1) *Let (M, g_0) be a compact Kähler manifold with $c_1 < 0$. Then there is a Kähler-Einstein metric $g : \rho_g = -g$. Moreover, if $-\gamma_{g_0}$ represents c_1, then g is the unique such metric cohomologous to g_0.*

In applications one uses the Kähler-Einstein metric as a canonical metric to simplify various formulas and give clearer insight into problems.

For example, if M is a compact Kähler surface (i.e. $\dim_{\mathbb{C}} M = 2$), then one can prove that the Chern classes satisfy $3c_2(M) \geq c_1^2(M)$, with equality if and only if M is biholomorphically covered by the ball in $\mathbb{C}^2$ (this was first proved by S. T. Yau, although the inequality $3c_2 \geq c_1^2$ for Kähler-Einstein metrics had been observed by H. Guggenheimer in 1952). To give the proof, one uses the known fact that the characteristic class $3c_2(M) - c_1^2(M)$ can be written as a complicated integral involving curvature

$$3c_2 - c_1^2 = \int_M (\text{curvature terms}).$$

Since the left is independent of the Kähler metric, we may use any convenient metric. In particular, if we use a Kähler-Einstein metric, the integrand is simply a square so the inequality $3c_2 - c_1^2 \geq 0$ becomes evident. It is also easy to check when equality can occur. (An analogous proof in two real dimensions is to obtain the sign of the Euler characteristic, $\chi(M)$, using the Gauss-Bonnet theorem and the existence of constant curvature metrics).

d) Complex Monge-Ampère equations: Existence

Both the Calabi problem and the existence of Kähler-Einstein metrics lead us to solve the complex Monge-Ampère equation

$$\det(g_0 + \varphi'') = (\det g_0) e^{f - \lambda\varphi}, \tag{9.34}$$

requiring that $g_0 + \varphi''$ be definite. If $\lambda = 0$, we must add the necessary condition (9.29).

We will sketch the existence proofs. They use the continuity method. The easier case is $\lambda < 0$.

$\lambda < 0$. Consider the family of problems

$$\det(g_0 + \varphi'') = (\det g_0) e^{tf - \lambda\varphi}, \qquad 0 \leq t \leq 1 \tag{9.35}$$

where $g_0 + \varphi''$ is positive definite. At $t = 0$ we have the obvious solution $\varphi = 0$. Let A be the subset of $t \in [0, 1]$ such that one can solve (9.35). To prove that A is open is a routine application of the implicit function theorem (an exercise for the reader).

The proof that A is closed is, as usual, more difficult. The first step is to obtain a uniform estimate for the solution, independent of $t \in [0, 1]$. At the point where φ has its maximum, we use local coordinates in which g_0 is the identity and φ'' is diagonal with non-positive eigenvalues $\alpha_k \leq 0$ (because φ has its maximum there). Then (9.5) reads

$$1 \geq \Pi_k(1 + \alpha_k) = e^{tf - \lambda\varphi}, \qquad 0 \leq t \leq 1$$

so

$$|\lambda|\varphi \leq \max_M f.$$

By also considering the point where φ has its minimum we obtain the uniform estimate

$$|\varphi| \le \|f\|_\infty / |\lambda| \tag{9.36}$$

To estimate the higher derivatives of φ we must work harder. We estimate $\Delta\varphi$ [It would be nice if there were some general procedure for doing this, as we did in Chapter 5.3, but there is none yet]. Let

$$F = \log(m - \Delta_K \varphi) - c\varphi,$$

where c is a sufficiently large real constant, and let Δ'_K be the Laplacian in the metric $g = g_0 + \varphi''$. At the point where F has its maximum, then clearly $\Delta'_K F \le 0$. A computation using this and (9.36) shows that for some constant c,

$$0 < m + \Delta_K \varphi \le c \tag{9.37}$$

(the inequality $0 < m + \Delta_K \varphi$ is obvious since $g = g_0 + \varphi''$ is positive definite and $m = \dim_{\mathbb{C}} M$). This gives a uniform estimate on $\Delta_K \varphi$. It also shows that for $0 \le t \le 1$ all of the metrics $g = g_0 + \varphi''$ are uniformly equivalent (to prove this, use local coordinates in which g_0 is the identity and φ'' is diagonal at the point $z \in M$ under consideration).

Next one estimates the Hölder norm $\|\varphi\|_{C^{2,\sigma}}$. Here one can apply a general result of Evans (see [GT, second edition] for a simplified proof). This approach replaces a special and complicated estimate of the third derivatives for (9.35). Once one has a $C^{2,\sigma}$ *a priori* estimate for the solutions of (9.35), then one can estimate the third derivatives of φ by differentiating the equation (9.35) and observing that the first derivatives of φ satisfy a *linear* equation whose coefficients we have just estimated. Repeating this one can estimate all the derivatives of φ that one wishes.

Using these estimates for φ and its derivatives, the standard procedure of Chapter 5.3 show that the set A is closed and hence that for $\lambda < 0$, there is a (unique) solution φ of (9.34) with $g_0 + \varphi''$ positive definite.

For more details of this proof (as well as for related facts) see [Au-4] and [SP], except see [GT] for the Hölder estimate on the second derivatives.

$\lambda = 0$. In view of the necessary condition (9.29) we consider the equation

$$\det(g_0 + \varphi'') = (\det g_0)[1 + t(e^f - 1)] \tag{9.38}$$

(one can devise many equally suitable equations). Note that $g := g_0 + \varphi''$ is required to be positive definite. Write (9.38) as

$$F(\varphi) = 1 + t(e^f - 1). \tag{9.39}$$

Let $Q_c^{k,\sigma} = \left\{ u \in C^{k,\sigma} \text{ with } \int_M u \, dv_{g_0} = c \right\}$. Then

$$F : Q_0^{k+2,\sigma} \to Q_{\mathrm{Vol}(M,g_0)}^{k+2,\sigma} \tag{9.40}$$

For the continuity method, to prove the openness at t_0, let φ be the solution at t_0. Then by a computation

$$F'(\varphi)\psi = F(\varphi)\Delta'_K \psi,$$

where $\Delta'_K \varphi$ is the Laplacian in the metric $g' = g_0 + \varphi''$. By (9.23)

$$\int F(\varphi)\Delta'_K \psi \, dv_{g_0} = \int \Delta'_K \psi \, dv_g = 0.$$

Thus the linearization, $F'(\varphi)$, is an isomorphism between the tangent spaces of the spaces in (9.40). This proves the openness.

For the closedness, we again need *a priori* estimates. In the case $\lambda < 0$ the uniform estimate was a simple consequence of the maximum principle. For the present case, $\lambda = 0$, a totally different and much more difficult procedure is required. At the present time, the simplest procedure is to obtain an L^p estimate of solutions of (9.38) of the form

$$(\int_M |\varphi|^p \, dv_{g_0})^{1/p} \le c_p, \tag{9.41}$$

where c_p is independent of t and $\lim c_p = c < \infty$ as $p \to \infty$. Thus, letting $p \to \infty$ in (9.41) we obtain the estimate: $\max|\varphi| \le$ constant. From here on, one uses the same estimates already discussed for the case $\lambda < 0$ on the higher derivatives of φ to complete the *a priori* estimates and hence the proof. Again, see [Au-4], [SP] and [GT] for more details.

Appendix: Some Geometry Formulas

[*needs substantial revision*]

The primary purpose of collecting these formulas here is to fix our notation and sign conventions.

Linear Algebra We begin with two basic formulas that are often neglected in elementary courses. Let $A(t)$ be an invertible matrix whose elements depend smoothly on the real parameter t.

Derivative of $A^{-1}(t)$.

$$\frac{dA^{-1}(t)}{dt} = -A^{-1}(t)\frac{dA(t)}{dt}A^{-1}(t) \tag{A.1}$$

To prove this, differentiate the identity $A(t)A^{-1}(t) = I$ to find $A'A^{-1} + AA^{-1\prime} = 0$. Solving this for $A^{-1\prime}$ gives the result.

Derivative of $\det A(t)$. If $A(t)$ is invertible then

$$\frac{d\det A(t)}{dt} = \det A(t)\ \mathrm{trace}\Big(A^{-1}(t)\frac{dA(t)}{dt}\Big) \tag{A.2}$$

It is enough to verify this at $t = 0$. First in the special case of a matrix $B(t)$ with $B(0) = I$ we have $B(t) = I + C(t)t$, where $C(0) = B'(0)$. Then

$$\det B(t) = 1 + [\mathrm{trace}\, C(0)]t + o(t) = 1 + [\mathrm{trace}\, B'(0)]t + o(t)$$

and the result is clear.

We can reduce to the special case by writing $A(t) = A(0)B(t)$. Then observe that $B(0) = I$. Therefore

$$\frac{d\det A(t)}{dt}\Big|_0 = \det A(0)\ \frac{d\det B(t)}{dt}\Big|_0.$$

But $B'(0) = A^{-1}(0)A'(0)$, from which the result is clear.

Riemannian Metric and Geodesics Let M^n be a smooth (C^∞) n-dimensional manifold with tangent bundle TM and let $S(TM)$ denote the set of smooth vector fields on M. In local coordinates $(x^1, \ldots, x^n)$ then $dx^1, \ldots, dx^n$ are a basis for the differential *1*-forms and $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ are a dual basis for the vector fields.

Riemannian Metrics A *Riemannian metric is a positive definite quadratic form* g that defines an inner product on vector fields V, W

$$\langle V, W\rangle = g(V, W). \tag{A.3}$$

Consequently, in local coordinates the Riemannian metric is given by the positve definite (symmetric) matrix

$$g_{ij} = \Big\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big\rangle, \tag{A.4}$$

and we write

$$g = ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)\, dx^i\, dx^j = g_{ij}(x)\, dx^i\, dx^j, \tag{A.5}$$

where on the right we used the *summation convention* of summing on repeated indices. We will sometimes use this convention. The above formula also introduces the usual element of *arc length*, ds. It is also standard to write g^{ij} for the inverse of the matrix g_{ij} and $|g| = \det g$ so the *Riemannian element of volume* is

$$dx_g = \sqrt{|g|}\, dx_1 \cdots dx_n = \sqrt{|g|}\, dx.$$

If in local coordinates $V = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ and $W = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i}$, then from (A.3)–(A.4) the inner product is

$$\langle V,\, W\rangle = g_{ij} v^i w^j.$$

For brevity in local coordinates it is customary to write a vector field V simply as v^i.

In carrying out computations, it is useful to know that one can always introduce coordinates with the properties that at *one* point p one has

$$g_{ij}(p) = \delta_{ij} \qquad \text{and, for all} \quad i, j, k \qquad \frac{\partial g_{ij}(x)}{\partial x^k}\Big|_{x=p} = 0. \tag{A.6}$$

Indeed, one can obtain this by a change of coordinates which is a polynomial of degree at most two - but a more conceptual approach is to use so-called Riemannian *normal coordinates*, where one uses as local coordinates at the given point the geodesics (see below) which start at the given point; actually, these geodesics give polar coordinates, from which one obtains a set of corresponding cartesian coordinates by the usual formulas. One obvious advantage of (A.6) is that the Christoffel symbols (see below) are then zero at this one point.

Although one can pick coordinates so the first derivatives of the metric zero at one point, there are essential obstructions to making the second derivatives of the metric zero. These obstructions are measured by the curvature, which will be introduced below. Riemann showed that one can introduce local coordinates so that a metric is the standard Euclidean metric if and only if the curvature is zero (see the extensive discussion in Spivak [Sp, Vol. 2], where this is called the "Test Case"). Riemann also used an enlightening counting argument. He pointed out that locally on an n-dimensional manifold a metric is a symmetric matrix and so has $\frac{1}{2}n(n+1)$ functions. However a change of coordinates allows one to impose at most n conditions on the metric. Thus he states that there should be some set of $\frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1)$ functions which determine a metric. These functions are the curvature tensor (again, see [Sp, Vol 2] for more).

Since any inner product induces an identification between a vector space and its dual space, the Riemannian metric induces a natural identification between 1-forms and vector fields. In our local coordinates, the *1*-form v is $v = \sum v_i\, dx^i$; the dual vector field then has the coordinates $v^i = g^{ij} v_j$. Similarly $v_i = g_{ij} v^j$. This natural identification gives an inner product on *1*-forms α and β:

$$\langle \alpha,\, \beta\rangle = g^{ij}\alpha_i\beta_j = \alpha^i\beta_i.$$

GRADIENT Given a smooth function f, if we write $f_i = \frac{\partial f}{\partial x^i}$ then the differential of f is the *1*-form $df = f_i dx^i$ and its *gradient* is the dual vector field $\nabla f = \operatorname{grad} f = f^i = g^{ij} f_j$. Thus, as an exercise in notation, for functions φ and ψ we have

$$\langle \nabla\varphi,\, \nabla\psi\rangle = g_{ij}\varphi^i\psi^j = g^{ij}\varphi_i\psi_j = \varphi^i\psi_i.$$

In particular, $|\nabla\varphi|^2 = g^{ij}\varphi_i\varphi_j = \varphi^i\varphi_i$.

GEODESICS Knowing the element of arc length, we can determine the length of a curve. Given two points P and Q, it is natural to seek the shortest curve joining them. Such a curve is called a *geodesic*. In Euclidean space these are simply straight lines. Our discussion will be formal, presuming there is a smooth shortest curve $x(t) = (x^1(t), \dots, x^n(t))$ and that all discussion takes place in one coordinate patch in Euclidean space. We obtain the standard Euler-Lagrange differential equations in the calculus of variations.

Because arc length is independent of the parameterization, any parameterization of this shortest curve will be adequate. To simplify the computation we will assume the desired geodesic x is parametrized by arc length s. Then

$$\sum_{i,j} g_{ij}(x)\,\frac{dx^i}{ds}\frac{dx^j}{ds} = 1 \tag{A.7}$$

Say the curve has length L, $x(0) = P$ and $x(L) = Q$. Let $y(s) = (y^1(s), \dots, y^n(s))$ be a smooth curve with $y(0) = 0$ and $y(L) = 0$; of course the parameter s is not necessarily the arc length for the curve y. For all small real λ we consider the family of curves $x+\lambda y$ which also join P and Q. Then, using the notation $\dot{x} = dx/ds$, the arc length is given by

$$\mathcal{L}(x+\lambda y) = \int_0^L \sqrt{\sum_{i,j} g_{ij}(x+\lambda y)(\dot{x}^i + \lambda\dot{y}^i)(\dot{x}^j + \lambda\dot{y}^j)}\; ds$$

Since x is the shortest curve we see that the scalar-valued function $\varphi(\lambda) = \mathcal{L}(x+\lambda y)$ has a minimum at $\lambda = 0$. Thus by calculus $\frac{d\varphi}{d\lambda} = 0$ at $\lambda = 0$. We use the above formula for the length $\mathcal{L}$ along with the normalization (A.7) to compute this derivative:

$$0 = \frac{d}{d\lambda}\mathcal{L}(x+\lambda y)\Big|_{\lambda=0} = \int_0^L \Big[\sum_{i,j} g_{ij}(x)\dot{x}^i\dot{y}^j + \tfrac{1}{2}\sum_{i,j,h}\frac{\partial g_{ij}(x)}{\partial x^h}y^h\dot{x}^i\dot{x}^j\Big]\,ds$$

Now integrate the first term by parts, removing the derivative from the $\dot{y}^j$, observing that there are no boundary terms since $y(0) = y(L) = 0$, to find

$$0 = \int_0^L \sum_h \Big[-\sum_i \frac{d}{ds}\big(g_{ih}(x)\dot{x}^i\big) + \tfrac{1}{2}\sum_{i,j}\frac{\partial g_{ij}(x)}{\partial x^h}\dot{x}^i\dot{x}^j\Big]y^h\,ds.$$

Since the y^h are arbitrary except for their boundary values, we deduce that the remaining term in the integrand must be zero. We rewrite this, using summation convention, as the following Euler-Lagrange differential equation for the problem of minimizing the arc length, that is, for finding geodesics:

$$g_{ih}(x)\ddot{x}^i + \frac{\partial g_{ih}(x)}{\partial x^j}\dot{x}^i\dot{x}^j - \tfrac{1}{2}\frac{\partial g_{ij}(x)}{\partial x^h}\dot{x}^i\dot{x}^j = 0,$$

Equivalently, we can rewrite this as

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \tag{A.8}$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{kh}\Big(\frac{\partial g_{hj}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^h}\Big). \tag{A.9}$$

The Γ^k_{ij} are called the *Christoffel symbols* associated with the metric. Since the coefficient of Γ^k_{ij} in (A.8) is symmetric in $i\ j$, we have also defined Γ^k_{ij} to have the same symmetry: $\Gamma^k_{ij} = \Gamma^k_{ji}$. When we define the covariant derivative below we will again obtain the Christoffel symbols and there this symmetry will be more natural. From the usual existence theorem for ordinary differential equations, we can find a unique solution with given initial position and tangent vector. Note that in this brief discussion of geodesics, we only used the first derivative of the arc length functional. Thus, we have not at all considered the issues of when a geodesic actually minimizes the distance. As one might surmise, a treatment of this involves the second derivative of the arc length functional.

Connection, Covariant Derivative

Covariant Derivative of a Vector Field On an arbitrary manifold, there is no invariant way to take the derivative of a vector field A *connection* gives a rule for differentiating a vector field (one can equivalently view it as a way of defining parallel translation of a vector, although we shall not take the time to do so here). On a Riemannian manifold, the metric itself defines an inner product and hence specifies how to translate a vector field parallel to itself so one can use this to define the derivative as the limit of a difference quotient. We will take a different approach to defining the derivative. A connection defines an operator $\nabla : S(TM)\times S(TM)\to S(TM)$. One thinks of $\nabla_V W$ as the directional derivative of W in the direction of V. The Riemannian connection has the following properties for all vector fields $V, W, Z \in S(TM)$ and all $\varphi,\psi\in C^\infty(M)$:

1) $\nabla_{\varphi V+\psi W}Z = \varphi\nabla_V Z + \psi\nabla_W Z$,
2) $\nabla_V(\varphi W+\psi Z) = V(\varphi)W + \varphi\nabla_V W + V(\psi)Z + \psi\nabla_V Z$,
3) $V\langle W,\, Z\rangle = \langle\nabla_V W,\, Z\rangle + \langle W,\, \nabla_V Z\rangle$ (*compatible with the metric*),

and

4) $\nabla_V W - \nabla_W V = [V,W]$ (connection is *torsion free*).

Given a metric, one proves there is a unique connection with these properties. One way to do this is to give an explicit (messy) formula for the connection using the metric. In local coordinates, since $\nabla_{\frac{\partial}{\partial x^i}}\big(\frac{\partial}{\partial x^j}\big)$ is a vector field, it is some linear combination of the basis vectors $\frac{\partial}{\partial x^i}$. One writes

$$\nabla_{\frac{\partial}{\partial x^i}}\big(\tfrac{\partial}{\partial x^j}\big) = \sum \Gamma^k_{ij}\,\tfrac{\partial}{\partial x^k}. \tag{A.10}$$

The torsion-free Property 4 gives the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$. In view of the linearity Properties 1-2, the coefficients Γ^k_{ij}, called the *Christoffel symbols*, define the connection $\nabla_V W$ for any vector fields V and W. The Christoffel symbols will turn out to be the same as thse used above in our discussion of geodesics.

We now use Property 3 with equation (A.4) to compute these Christoffel symbols. First

$$\frac{\partial g_{ij}}{\partial x^h} = \frac{\partial}{\partial x^h}\Big\langle\frac{\partial}{\partial x^i},\,\frac{\partial}{\partial x^j}\Big\rangle = \sum_s \Gamma^s_{hi}\,g_{sj} + \sum_s \Gamma^s_{hj}\,g_{is}. \tag{A.11}$$

Using this formula one computes $\frac{\partial g_{hj}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^h}$. After observing the cancellation one again obtains the standard formula (A.9) for the Christoffel symbols.

Using Property 2 and equation (A.10) it is now straightforward to compute the covariant derivative of a vector field $W = w^i\frac{\partial}{\partial x^i}$

$$\nabla_{\frac{\partial}{\partial x^j}}W = \frac{\partial w^i}{\partial x^j}\frac{\partial}{\partial x^i} + w^i\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} = \Big(\frac{\partial w^i}{\partial x^j} + w^k\Gamma^i_{kj}\Big)\frac{\partial}{\partial x^i} = w^i{}_{;j}\,\frac{\partial}{\partial x^i}, \tag{A.12}$$

where we have introduced the tensor notation

$$w^i{}_{;j} = w^i{}_{,j} + w^k\Gamma^i_{kj}, \tag{A.13}$$

with $w^i{}_{,j} = \frac{\partial w^i}{\partial x^j}$. If one uses this notation, the subtle visual difference between $w^i{}_{;j}$ and $w^i{}_{,j}$ can cause difficulties. Note that for a scalar-valued function φ the covariant derivative and ordinary partial derivatives are the same so in tensor notation $d\varphi = \varphi_{;i}\,dx^i = \varphi_{,i}\,dx^i$, that is, $\varphi_{;i} = \varphi_{,i}$.

If $V = \sum v^i \frac{\partial}{\partial x^i}$, one can clearly compute $\nabla_V W$ using the linearity property 1) and (A.13).

Implicit in the above discussion was the following definition of ∇:

$$\nabla W(V) = \nabla_V W \tag{A.14}$$

so, for example,

$$\nabla W\left(\tfrac{\partial}{\partial x^i}\right) = w^k_{,i}\tfrac{\partial}{\partial x^i}$$

and hence

$$\nabla W = \sum_{i,k} w^k{}_{;i}\,dx^k \otimes \tfrac{\partial}{\partial x^i}. \tag{A.15}$$

This explicitly exhibits ∇W in terms of its classical tensor components $w^k{}_{;i}$. It also shows that ∇W is a tensor of type $(1,\,1)$ (it has one "upper" index and one "lower" index), while W is a tensor of type $(1,\,0)$.

As a digression we apply equation (A.2) to record the following formula for $\sum_j \Gamma^j_{ij}$. We will use it shortly.

$$\sum_j \Gamma^j_{ij} = \tfrac{1}{2}\sum_{jk} g^{jk}\frac{\partial g_{jk}}{\partial x^i} = \frac{1}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial x^i} \tag{A.16}$$

Other Covariant Derivatives One uses the covariant derivative of a vector field to define the covariant derivative of all other tensors. Instead of a general definition, we give a few examples from which the general case should be clear. For any function f we define $\nabla_V f = V(f)$. We assume the usual product rule holds for covariant differentiation of any tensor field as well as the linearity modeled on previous Properties 1) and 2) for the covariant derivative of a vector field.

We first treat differential 1-forms. Let $\alpha = \alpha_i\,dx^i$ be a 1-form. We compute $\nabla_V\alpha$ which will be another 1-form. Say $\nabla_{\frac{\partial}{\partial x^j}}\alpha = \alpha_{i;j}\,dx^i$. We wish to compute the $\alpha_{i;j}$. The key procedure is to introduce a vector field $W = w^i\frac{\partial}{\partial x^i}$ and use that $\alpha(W) = \alpha_i w^i$ is a scalar-valued function. Then using these rules we find the formula by the following sequence of steps (the left side of the second equation follows from the left side of the first equation etc.).

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^j}}(\alpha(W)) &= (\nabla_{\frac{\partial}{\partial x^j}}\alpha)(W) + \alpha(\nabla_{\frac{\partial}{\partial x^j}}W)\\
\nabla_{\frac{\partial}{\partial x^j}}(\alpha_i w^i) &= (\alpha_{i;j}\,dx^i)\Big(w^k\frac{\partial}{\partial x^k}\Big) + (\alpha_i\,dx^i)\Big(w^k{}_{;j}\frac{\partial}{\partial x^k}\Big)\\
\alpha_{i,j}w^i + \alpha_i w^i{}_{,j} &= \alpha_{i;j}w^i + \alpha_i(w^i{}_{,j} + w^\ell\Gamma^i_{\ell j}).
\end{aligned}$$

After cancelling two terms and renaming some indices we are left with

$$\alpha_{i;j}w^i = \alpha_{i,j}w^i - \alpha_k w^i\Gamma^k_{ij}.$$

Since this must hold for *any* vector field W, we conclude that

$$\alpha_{i;j} = \alpha_{i,j} - \alpha_k\Gamma^k_{ij}, \tag{A.17}$$

that is,

$$\nabla_{\frac{\partial}{\partial x^j}} \alpha = \left(\frac{\partial \alpha_i}{\partial x^j} - \alpha_k \Gamma^k_{ij} \right) dx^i.$$

Just as in the case of vector fields, this leads us to write

$$\nabla \alpha = \sum_{ij} \alpha_{i;j} \, dx^i \otimes dx^j. \tag{A.18}$$

We next show that the metric itself has covariant derivative zero, $\nabla_V g = 0$. As we saw above, one can simplify computations by working directly with the coefficients of the tensors. Using our assumption that the product the product rule holds for covariant differentiation, we see that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^k}} \langle W, Z \rangle &= \nabla_{\frac{\partial}{\partial x^k}} (g_{ij} W^i Z^j) \\ &= (\nabla_{\frac{\partial}{\partial x^k}} g_{ij}) \, W^i Z^j + g_{ij} (\nabla_{\frac{\partial}{\partial x^k}} W^i) \, Z^j + g_{ij} W^i \, (\nabla_{\frac{\partial}{\partial x^k}} Z^j) \\ &= (\nabla_{\frac{\partial}{\partial x^k}} g_{ij}) \, W^i Z^j + \langle \nabla_{\frac{\partial}{\partial x^k}} W, \ Z \rangle + \langle W, \nabla_{\frac{\partial}{\partial x^k}} Z \rangle. \end{aligned}$$

But by property 3) of the covariant derivative (compatibility with the metric), this is equal to the same right-hand-side *without* the first term. Thus $\nabla_{\frac{\partial}{\partial x^k}} g_{ij} = 0$ and hence, by linearity, $\nabla_V g = 0$ for any vector field V. In tensor notation, $g_{ij;k} = 0$. After one knows how to compute $h_{ij;k}$, that is, the covariant derivative of any tensor field of the form $h_{ij} \, dx^i \otimes dx^j$, then a special case is the computation showing that $g_{ij;k} = 0$.

We next compute the second derivative $\nabla^2 W$ of a vector field W. The result, defined below, will be a tensor field of type $(1, 2)$. Now since $\nabla_Z W$ is itself a vector field, we can compute $\nabla_V (\nabla_Z W)$. However, using the definition (A.14) and the assumption that the product rule holds for differentiation:

$$\nabla_V [\nabla_Z W] = \nabla_V [\nabla W(Z)] = (\nabla_V \nabla W)(Z) + \nabla W(\nabla_V Z) \tag{A.19}$$

$$= (\nabla_V \nabla W)(Z) + \nabla_{\nabla_V Z} W. \tag{A.20}$$

This formula defines $(\nabla_V \nabla W)(Z)$. Also, as in (A.14), we define $\nabla^2 W$ to be

$$\nabla^2 W(Z, V) = (\nabla_V \nabla W)(Z).$$

Combining the last two formulas we find that

$$\nabla^2 W(Z, V) = \nabla_V [\nabla_Z W] - \nabla_{\nabla_V Z} W. \tag{A.21}$$

The same procedure works for the second derivative of any tensor field.

We now carry out the straightforward, although tedious, details to write this in local coordinates for $V = \frac{\partial}{\partial x^k}$ and $Z = \frac{\partial}{\partial x^j}$. Since $\nabla_{\frac{\partial}{\partial x^j}} W = w^i{}_{;j} \frac{\partial}{\partial x^i}$, we have

$$\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} W = w^i{}_{;j,k} \frac{\partial}{\partial x^i} + w^i{}_{;j} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = \left(w^i{}_{;j,k} + w^s{}_{;j} \Gamma^i_{sk} \right) \frac{\partial}{\partial x^i}. \tag{A.22}$$

Substituting the formula (A.13) for $w^i{}_{;j}$ we get the long formula

$$\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} W = \left(w^i{}_{,jk} + w^\ell{}_{,k} \Gamma^i_{\ell j} + w^\ell{}_{,j} \Gamma^i_{\ell k} + w^\ell \frac{\partial \Gamma^i_{\ell j}}{\partial x^k} + w^\ell \Gamma^s_{\ell j} \Gamma^i_{sk} \right) \frac{\partial}{\partial x^i}. \tag{A.23}$$

This formula reveals important information. It is not symmetric in j and k, so second derivatives do not commute, but since the first three terms on the right are symmetric in j and k, the error does not involve any derivatives of the vector field W. We will shortly use this observation to define the curvature.

HESSIAN, LAPLACIAN, AND DIVERGENCE The Hessian of a function φ is defined as

$$\operatorname{Hess}\varphi = \varphi_{;ij} = \varphi_{,ij} - \varphi_{,k}\Gamma^k_{ij}. \tag{A.24}$$

Note that this is symmetric in ij and defines a quadratic form on vector fields.
Using formula A.16 we have the following formulas for the Laplacian

$$\Delta\varphi = g^{ij}\varphi_{;ij} = \varphi_{;i}{}^{i} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\Big(g^{ij}\sqrt{|g|}\frac{\partial\varphi}{\partial x^j}\Big), \tag{A.25}$$

Note that, for good reason, many mathematicians use the *opposite* sign for the Laplacian.

We will define the *divergence* of a vector field V so that the Divergence Theorem is valid. The technique we use is both simple and useful. For a bounded region Ω with smooth boundary the Divergence Theorem states that

$$\int_\Omega \operatorname{div} V\, dx_g = \int_{\partial\Omega} V\cdot N\, dA_g, \tag{A.26}$$

where N and dA_g are the unit outer normal vector and element of volume on the boundary. The inner product on the right is in the Riemannian metric. We also require the property (derivation) that for any scalar-valued function φ

$$\operatorname{div}(\varphi V) = \nabla\varphi\cdot V + \varphi\operatorname{div} V,$$

where again the inner product in the first term on the right is in the Riemannian metric. To avoid worrying about the boundary term in the Divergence theorem, we will use functions φ whose support lies inside Ω and that Ω lies in a coordinate patch so that we can apply the classical form of the Divergence theorem. First observe that the above formula for the volume element dx_g and an ordinary integration by parts and give

$$\int_\Omega \nabla\varphi\cdot V\, dx_g = \int_\Omega \frac{\partial\varphi}{\partial x^i}v^i\sqrt{|g|}\, dx = -\int_\Omega \varphi\frac{\partial(\sqrt{|g|}\, v^i)}{\partial x^i}\, dx$$

Using this in the Divengernce Theorem we find

$$0 = \int_\Omega \operatorname{div}(\varphi V)\, dx_g = \int_\Omega \varphi\Big(-\frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}\, v^i)}{\partial x^i} + \operatorname{div} V\Big)\, dx_g$$

Because this is to hold for any smooth φ, we obtain the desired formula for the divergence of a vector field,

$$\operatorname{div} V = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}\, v^i)}{\partial x^i}$$

Again we caution that many mathematicians use the opposite sign for the divergence. From equations (A.16) and (A.12) we can also write the divergence using vector field notation as

$$\operatorname{div} V = \sum_i \langle\nabla_{e_i}V,\, e_i\rangle.$$

In classical vector analysis one often writes $\operatorname{div} V = \nabla\cdot V$; despite temptations we have avoided this because of possible confusion with the covariant derivative. The divergence can also be used to write the Laplacian as $\Delta\varphi = \operatorname{div}\nabla\varphi$ and of course give the same formula (A.25). To many, including myself, this second approach to the Laplacian — using the Divergence Theorem — is more natural.

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Riemann curvature tensor

$$\operatorname{Riem}(g) = R^h{}_{ijk} = \frac{\partial\Gamma^h_{ik}}{\partial x^j} - \frac{\partial\Gamma^h_{ij}}{\partial x^k} + \sum_\ell\Big(\Gamma^h_{\ell j}\Gamma^\ell_{ik} - \Gamma^h_{\ell k}\Gamma^\ell_{ij}\Big) \tag{A.27}$$

Ricci curvature tensor

$$\begin{aligned}\operatorname{Ric}(g) = R_{ij} = R^h{}_{ihj} = &\tfrac{1}{2}g^{st}\Big(\frac{\partial^2 g_{is}}{\partial x^j\partial x^t} + \frac{\partial^2 g_{is}}{\partial x^i\partial x^t} - \frac{\partial^2 g_{ij}}{\partial x^s\partial x^t} - \frac{\partial^2 g_{st}}{\partial x^i\partial x^j}\Big)\\ &+ Q(g,\partial g),\end{aligned} \tag{A.28}$$

where Q is a function of g and its first derivatives only, and is homogeneous of degree 2 in the first derivatives, ∂g. For the standard sphere (S^n, g_0) of radius 1 in $\mathbb{R}^{n+1}$

$$\operatorname{Ric}(g_0) = (n-1)g_0.$$

Scalar curvature

$$S(g) = S = g^{ij}R_{ij} = R_i{}^i \tag{A.29}$$

so on (S^n, g_0), $S(g_0) = n(n-1)$. If $\dim M = 2$, then Gauss curvature $= \frac{1}{2}$ scalar curvature.

Second Bianchi identity

$$R_{tijk;\ell} + R_{ti\ell j;k} + R_{tik\ell;j} = 0, \tag{A.30}$$

where the semi-colon ; is covariant differentiation. In particular, for the Ricci tensor this gives

$$2R_{ik;}{}^i = S_{;k} \tag{A.31}$$

First variation formulas

Let $g(t)$ be a family of metrics for real t, with $dg/dt|_{t=0} = h$. Then

$$\frac{d}{dt}\operatorname{Riem}(g(t))|_{t=0} = \tfrac{1}{2}g^{is}(h_{is;kj} + h_{ks;ij} - h_{ki;sj} - h_{is;jk} - h_{js;ik} + h_{ji;sk}), \tag{A.32}$$

where the covariant derivatives are in the $g(0)$ metric.

$$\frac{d}{dt}\operatorname{Ric}(g(t))|_{t=0} = \tfrac{1}{2}(h_i{}^\ell{}_{;k\ell} + h_k{}^\ell{}_{;i\ell} - h_{ki;\ell}{}^\ell + h_\ell{}^\ell{}_{;ik}) \tag{A.33}$$

$$\frac{d}{dt}S(g(t))|_{t=0} = -h_i{}^i{}_{;s}{}^s + h^{is}{}_{;is} - h^{is}R_{is} \tag{A.34}$$

Pointwise Conformal metrics on (M^n, g_0)

If $g_1 = e^{2u}g$, so g_1 is *pointwise conformal* to g, then

$$dx_{g_1} = e^{nu}dx_g, \tag{A.35}$$

$$|\nabla_{g_1}\varphi|^2 = e^{-2u}|\nabla\varphi|^2, \text{ for any function } \varphi, \tag{A.36}$$

$$\operatorname{Ric}(g_1) = \operatorname{Ric}(g) - (n-2)(u_{;ij} - u_{;i}u_{;j}) - g_{ij}(\Delta u + (n-2)|\nabla u|^2), \tag{A.37}$$

where Δ and ∇ are the Laplacian and gradient in the g metric.

$$S(g_1) = e^{-2u}[-2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + S(g)]. \tag{A.38}$$

If $n = 2$, then $S(g) = 2K(g)$, where $K(g)$ is the Gauss curvature of g so

$$K_{g_1} = e^{-2u}[-\Delta u + K] \tag{A.39}$$

Dimension 2: If $g = E du^2 + 2F\,du\,dv + G\,dv^2$ with

$$E(p) = G(p) = 1,\ F(p) = 0, \quad \text{and} \quad \nabla E(p) = \nabla F(p) = \nabla G(p) = 0 \tag{A.40}$$

at a point p, then

$$\text{Gauss curvature}\,(p) = K(p) = -\tfrac{1}{2}(E_{vv} - 2F_{uv} + G_{uu}), \tag{A.41}$$

where E_{vv} is the second partial derivative with respect to v, etc.

In particular, if

$$g_1 = g - (dz)^2, \tag{A.42}$$

where g is as above and $\nabla z(p) = 0$, then

$$K_1(p) = K(p) - (z_{uu}z_{vv} - {z_{uv}}^2). \tag{A.43}$$

This is a special case of the following formula for the Gauss curvature of $g_1 = g - (dz)^2$ at any point:

$$K_1 = \frac{K}{1 - |\nabla z|^2} - \frac{\det(\operatorname{Hess} z)}{(1 - |\nabla z|^2)^2 \det g} \tag{A.44}$$

Ricci commutation formulas

$$v_{i;k\ell} = v_{i;\ell k} - v_j R^j{}_{i\ell k} \qquad \text{and} \qquad v^i{}_{;k\ell} = v^i{}_{;\ell k} + v^j R^i{}_{j\ell k} \tag{A.45}$$

These (equivalent) formulas show that covariant derivatives commute except for a correction involving the curvature. They are often used as the definition of the Riemann curvature. They imply similar formulas for more complicated tensor, such as

$$h_{ij;k\ell} = h_{ij;\ell k} - h_{is}R^s{}_{j\ell k} - h_{sj}R^s{}_{i\ell k} \tag{A.46}$$

Weitzenböck formulas

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