Pythagorean Theorem: Using Similar Triangles

Let $T$ be a right triangle whose sides have length $a$, $b$, and $c$ ($c$ is the hypotenuse). The Pythagorean Theorem says that

$$a^2 + b^2 = c^2.$$  

This is Euclid’s proof using similar triangles. It also uses that for right triangles it is more informative to specify the sides than the vertices.

Let the right triangle $T'$ with sides $a'$, $b'$, $c'$ be similar to $T$. We then know the corresponding sides of $T$ and $T'$ are proportional, that is, there is a scaling factor $t > 0$ so that

$$a' = ta, \quad b' = tb, \quad c' = tc.$$

First step: Compare Area($T$) and Area($T'$). Because $T$ and $T'$ are right triangles,

$$\text{Area}(T') = \frac{1}{2}a'b' = \frac{1}{2}(ta)(tb) = t^2 \text{Area}(T).$$

In our situation (below) we will know the hypotenuses $c$ and $c'$ so $t = \frac{c'}{c}$ and

$$\text{Area}(T') = \left[\frac{\text{hypotenuse}(T')}{\text{hypotenuse}(T)}\right]^2 \text{Area}(T).$$

Now Euclid’s key idea: Introduce the altitude to the hypotenuse of $T$. This partitions $T$ into two triangles, $T_1$ and $T_2$. Both of them are similar to $T$ since their corresponding angles are equal. By comparing the length of the hypotenuse of $T$, $T_1$ and $T_2$ we find the scaling factors:

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>$T_1$</th>
<th>$T_2$</th>
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<tbody>
<tr>
<td>hypotenuse</td>
<td>$c$</td>
<td>$a$</td>
<td>$b$</td>
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Use equation (2) to find $\text{Area}(T_1) = (a/c)^2 \text{Area}(T)$ and $\text{Area}(T_2) = (b/c)^2 \text{Area}(T)$. But $\text{Area}(T) = \text{Area}(T_1) + \text{Area}(T_2)$ so

$$\text{Area}(T) = [(a/c)^2 + (b/c)^2] \text{Area}(T).$$

Dividing by Area($T$) gives exactly the Pythagorean equation (1).

It may be useful to compare this with other recent presentations. They involve more formulas – and motivated this version.

WIKIPEDIA:  
(search for "Proof using similar triangles")

Khan Academy:  
https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-pythagorean-proofs/v/pythagorean-