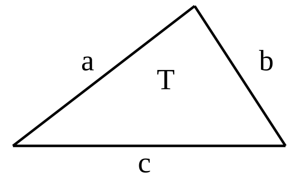
Math 202 Jerry L. Kazdan

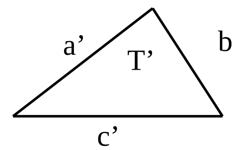
Pythagorean Theorem: Using Similar Triangles

Let T be a right triangle whose sides have length a, b, and c (c is the hypotenuse). The Pythagorean Theorem says that



$$a^2 + b^2 = c^2. \tag{1}$$

This is Euclid's proof using similar triangles. I wrote it because I was unhappy with what I found on the Internet.



The right triangle T' with sides a', b', c' is similar to T because the corresponding angles are equal. We then know the corresponding sides of T and T' are proportional, that is, there is a *scaling factor* $t > 0$ so that

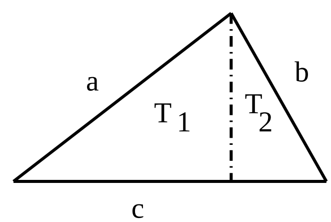
$$a' = ta, \quad b' = tb, \quad c' = tc.$$

First step: Compare Area(T) and Area(T'). Because T and T' are right triangles,

$$\text{Area}(T') = \tfrac{1}{2}a'b' = \tfrac{1}{2}(ta)(tb) = t^2\text{Area}(T).$$

In our situation (below) we will know the hypotenuses c and c' so $t = \frac{c'}{c}$ and

$$\text{Area}(T') = \left[\frac{\text{hypotenuse}(T')}{\text{hypotenuse}(T)}\right]^2 \text{Area}(T). \tag{2}$$



Now the key idea (Euclid!): Introduce the altitude to the hypotenuse of T. This partitions T into two triangles, T_1 and T_2. Both of them are similar to T since their corresponding angles are equal. By comparing the length of the hypotenuse of T, T_1 and T_2 we find the scaling factors:

	T	T_1	T_2
hypotenuse	c	a	b

Use equation (2) to find $\text{Area}(T_1) = (a/c)^2\text{Area}(T)$ and $\text{Area}(T_2) = (b/c)^2\text{Area}(T)$. But $\text{Area}(T) = \text{Area}(T_1) + \text{Area}(T_2)$ so

$$\text{Area}(T) = \left[(a/c)^2 + (b/c)^2\right]\text{Area}(T).$$

Dividing by Area(T) gives exactly the Pythagorean equation (1).

It may be useful to compare this with other recent presentations. They involve more formulas – and motuvated me to write this version.

Wikipedia:
`https://en.wikipedia.org/w/index.php?title=Pythagorean_theorem`
(search for "Proof using similar triangles")

Khan Academy:
`https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-pythagorean-proofs/v/pythagorean-`