Homework Set 2

[Submit any 5 problems. Due: Wednesday 30 Jan. in class.]

1. Let S^1 be the circle so $f \in C^1(S^1)$ means that f and its first derivative are both continuous and periodic with period 2π. Write the Fourier series as $f(x) = \sum_k f_k \frac{e^{ikx}}{\sqrt{2\pi}}$, so the Fourier coefficients are f_k.
   a) If $f \in \mathbb{C}^1(S^1)$, show that for some constant c we have $|f_k| \le \frac{c}{|k|}$.
   b) If $|f_k| \le \frac{c}{|k^2|}$, show that $f \in C(S^1)$.
   c) Show that $f \in C^\infty(S^1)$ if and only if for every $s \ge 0$ there is a constant $c(s)$ so that for all k one has $|f_k| \le \frac{c(s)}{(1+|k|)^s}$.

2. Show that for any $f \in C^\infty(T^2)$ and any real γ there is a (unique) solution $u \in \mathbb{C}^1(T^2)$ of
$$u_x - \gamma u_y + u = f(x,y).$$
Moreover, show that this solution $u \in \mathbb{C}^\infty(T^2)$.

3. [Semi-infinite string] For $x > 0$ let $u(x,t)$ be a solution of the wave equation with
$$u(x,0) = f(x), \quad u_t(x,0) = 0 \ \text{ for } \ x > 0, \qquad \text{while } \ u(0,t) = 0 \ \text{ for } \ t > 0.$$
Show that for $x > 0$ and $t > 0$
$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] & \text{for} \quad x > ct \\ \frac{1}{2}[f(ct+x) - f(ct-x)] & \text{for} \quad x < ct. \end{cases}$$
As an example, draw a sketch of the solution at $t = 0,\ 2,\ 4,\ 6,\ 8$ for the specific initial position
$$f(x) = \begin{cases} (x-2)(3-x)) & \text{for } 2 \le x \le 3, \\ 0 & 0 \le x \le 2 \text{ and } x > 3. \end{cases}$$

4. For the wave equation on the semi-infinite interval $0 < x$, solve the initial-boundary value problem if the end at $x = 0$ is free (Neumann boundary condition):
$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \ \text{ for } \ x > 0, \quad \text{while} \quad u_x(0,t) = 0 \ \text{ for } \ t > 0.$$

5. Find the motion $u(x,t)$ of a string $0 \le x \le \pi$ whose motion is damped:
$$u_{tt} + 2u_t = u_{xx},$$
with
$$u(x,0) = \sin 3x - 2\sin 5x, \quad u_t x, 0 = 0, \quad u(0,t) = u(\pi,t) = 0.$$

6. Let $x \in \mathbb{R}^3$. Maxwell's equations for an electromagnetic field $E(x,t) = (E_1, E_2, E_3)$, $B(x,t) = (B_1, B_2, B_3)$ in a vacuum are
$$E_t = \operatorname{curl} B, \quad B_t = -\operatorname{curl} E, \quad \operatorname{div} B = 0, \quad \operatorname{div} E = 0.$$
Show that each of the components E_j and B_j satisfy the wave equation $u_{tt} = u_{xx}$.

7. For a finite string $0 < x < L$ let u be a solution of the modified wave equation

$$u_{tt} + b(x,t)u_t = u_{xx} + a(x,t)u_x \tag{1}$$

with zero Dirichlet boundary conditions: $u(0,t) = u(L,t) = 0$, define the energy as

$$E(t) = \tfrac{1}{2}\int_0^L (u_t^2 + u_x^2)\,dx, \tag{2}$$

where we assume that $|a(x,t)|,\ |b(x,t)| < M$ for some constant M.
   a) Show that $E(t) \le e^{\alpha t}E(0)$ for some constant α depending only on M.
   b) What happens if you replace the Dirichlet boundary conditions by the Neumann boundary condition $\nabla u \cdot N = 0$ on the boundary (ends) of the string?
   c) Generalize part a) to a bounded region Ω in $\mathbb{R}^n$.

8. Let $u(x,t)$ be a solution of the wave equation (1) for $x \in \mathbb{R}$. Use an energy argument to show that the solution u has the same domain of dependence and range of influence as in the special case where $a(x,t) = b(x,t) = 0$.

9. Consider the equation

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0. \tag{3}$$

   a) Find a change of variable $\xi = ax + bt$, $\eta = cx + dt$ so that in the new coordinates the equation is the standard wave equation

$$u_{\xi\xi} = u_{\eta\eta}.$$

   b) Use this to solve (3) with the initial conditions

$$u(x,0) = x^2,\ u_t(x,0) = 2e^x.$$

10. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k},$$

where the coefficients a_{jk} are constants and (without loss of generality — why?) $a_{kj} = a_{jk}$. If the matrix $A = (a_{jk})$ is positive definite, show there is a change of variable $x = Sy$, where S is an $n \times n$ invertible matrix, so that in these new coordinates the equation becomes the standard wave equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\ell=1}^{n} \frac{\partial^2 u}{\partial^2 y_\ell}.$$

11. Let $x \in \mathbb{R}^n$.
   a) If function $w(x)$ depends only on the distance to the origin, $r = \|x\|$, show that

$$\Delta u = \frac{\partial^2 u}{\partial^2 r} + \frac{n-1}{r}\frac{\partial u}{\partial r}.$$

   b) Investigate solutions $u(x,t)$, $x \in \mathbb{R}^3$ of the wave equation $u_{tt} = \Delta u$ where $u(x,t) = v(r,t)$ depends only on r and t. For instance, are there solutions of the form $v(r,t) = \varphi(r)g(r-t)$?

[Last revised: April 3, 2008]