Some PDE Problems

1. a) Find a function $u(x,y)$ that satisfies $u_y = 1$ with $u(x,0)) = 2 + \sin 3x$.
   b) Find a function $u(x,y)$ that satisfies $u_x + u_y = 2u$ with $u(x,0)) = 2 + \sin 3x$ by first making an invertible change of variable of the form $s = ax + by$, $t = cx + dy$..
   c) Find a function $u(x,y)$ that satisfies $u_y = 2u$ with $u(x,0)) = 2 + \sin 3x$.

2. Let $u(x,y) = f(ax + by)$, where f is any smooth function. Show that $u_{xx}u_{yy} - u_{xy}^2 = 0$ for any contants a, b, c, and d.

3. For $-\infty < x < \infty$ and $t > 0$ consider the diffusion equation
$$u_t = u_{xx} + 2u_x + u \quad \text{with} \quad \mathrm{u}(x,0) = e^{-x^2} \tag{1}$$
Show that by making the change of variables of the form $u(x,t) = e^{ax+bt}v(x,t)$ using a clever choice of the constants a and b, the function v satisfies the standard diffusion equation
$$v_t = v_{xx}$$
but with a modified initial condition, $v(x,0)$.

4. Let $\mathbf{x}$ be a point in $\mathbb{R}^3$ and the vector field $\mathbf{F}(\mathbf{x})$ have continuous first derivatives. Assume $\mathbf{F}$ decays for large $|\mathbf{x}|$:
$$|\mathbf{F}(\mathbf{x})| \le \frac{1}{1 + |\mathbf{x}|^3}$$
for all $\mathbf{x}$. Show that
$$\iiint_{\mathbb{R}^3} \nabla \cdot \mathbf{F}\, dx\, dy\, dz = 0$$
Suggestion: First work with a large ball, $|\mathbf{x}| \le r$, apply the divergence theorem, and then let $r \to \infty$.

5. Consider the differential equation $\det\left(\dfrac{\partial^2 u(x)}{\partial x_i \partial x_j}\right) = 1$. Find solutions of the special form $u(x_1, \ldots, x_n) = f(r)$, where $r^2 = x_1^2 + \cdots + x_n^2$.

6. a) If $u_{xx} + u_{yy} \ge 0$ in a disc centered at p with radius a, show that in polar coordinates,
$$u(p) \le \frac{1}{2\pi} \int_0^{2\pi} u(a,\theta)\, d\theta.$$

b) Use this to show that if $\Delta u \geq 0$ in a bounded region $\mathcal{D} \subset \mathbb{R}^2$ with boundary $\mathcal{B}$, then

$$\max_{\mathcal{D}} u(x,y) \leq \max_{\mathcal{B}} u(x,y).$$

c) Apply this to show that if $\Delta v = f$ and $\Delta w = g$ with $f \geq g$ in $\mathcal{D}$ and $v = w$ on $\mathcal{B}$, then $v \leq w$ in $\mathcal{D}$.

7. Say $u(x,t)$ is a solution of the modified heat equation

$$u_t = u_{xx} - 2u_x \qquad \text{for } \ 0 < x < L \ \text{ and } \ t > 0 \tag{2}$$

.

a) If u satisfies the initial and boundary conditions

$$u(x,0) = 0, \quad \text{and} \ \ u(0,t) = 0, \quad u(L,t) = 0 \ \text{ for all } \ t \geq 0.$$

Use an "energy" argument involving

$$E(t) = \tfrac{1}{2} \int_0^L u^2(x,t)\, dx$$

to show that $u(x,t) = 0$ for all $t \geq 0$.

b) Show uniqueness of the initial/boundary value problem for solutions $w(x,t)$ of equation (2) that satisfy

$$w(x,0) = \varphi(x), \quad \text{and} \quad w(0,t) = g(t), \quad w(L,t) = h(t) \ \text{ for all } t \geq 0.$$

8. Let $f(x)$ and $g(x)$ be smooth functions that are periodic with period, say, 1. Investigate when one can find a periodic solution $u(x)$ with period 1 of the ordinary differential equation

$$u'' = f(x) - g(x)e^u.$$

Start by reducing to the case where $f(x)$ is a constant by letting w be a solution of $w'' = f - \bar{f}$, where $\bar{f}$ is the average of f over one period, and using the substitution $v = u - w$.

In particular, show that if $a > 0$ is a constant, then one can find a periodic solution of

$$u'' = a - h(x)e^u$$

if and only if the smooth periodic function h is positive somewhere.

9. On a compact Riemannian manifold $M^n(g)$ without boundary, let $a(x,s)$ be a smooth function, where $x \in M$ and $s \in \mathbb{R}$. If $|a(x,s)| \leq 1/2$, show that the PDE

$$\Delta u = a(x,u) + \cos u$$

has infinitely many distinct solutions. [Suggestion: Find appropriate sub and super solutions.]

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