

Curvature Functions for Compact 2-Manifolds

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Curvature functions for compact 2-manifolds*

By JERRY L. KAZDAN** and F. W. WARNER**

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I. Introduction

1. Three geometric problems

A basic problem in Riemannian geometry is that of describing the set of curvatures a given manifold can possess. In this paper we shall limit our discussion to compact, connected, two dimensional manifolds (not necessarily orientable). We will discuss open 2-manifolds and scalar curvatures for higher dimensional manifolds in separate papers [16, 17].

On a 2-manifold, there is essentially only one notion of curvature and our problem becomes that of describing the set of Gaussian curvature functions. There is only one known condition on curvature for compact 2-mani-

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folds — the global condition given by the Gauss-Bonnet theorem. If a smooth function K is the Gaussian curvature of some Riemannian metric \tilde{g} on M , then the Gauss-Bonnet theorem asserts that

$$(1.1) \quad \int_M K \, dA = 2\pi\chi(M) ,$$

where dA is the element of area with respect to \tilde{g} and $\chi(M)$ is the Euler characteristic of M . This clearly imposes the following sign conditions on K depending on $\chi(M)$:

- (a) $\chi(M) > 0$: K is positive somewhere ,
- (1.2) (b) $\chi(M) = 0$: K changes sign (unless $K \equiv 0$) ,
- (c) $\chi(M) < 0$: K is negative somewhere .

We were naturally led to ask the following, which can be thought of as a converse to the Gauss-Bonnet theorem.

QUESTION 1. *Are the sign conditions (1.2), depending on $\chi(M)$, sufficient conditions for a smooth function K on a compact 2-manifold to be the Gaussian curvature of some Riemannian structure on M ?*

To make the problem more tangible, we attempt to realize K in a very specific way. We attack Question 1 by prescribing some metric g on M , and by attempting to realize K as the curvature of a metric \tilde{g} that is conformally equivalent to g , or even more, of a metric that is pointwise conformal to g . (We shall call metrics \tilde{g} and g *pointwise conformal* if $\tilde{g} = e^{2u}g$ for some $u \in C^\infty(M)$, whereas we say that \tilde{g} and g are *conformally equivalent* if there is a diffeomorphism φ of M and a function $u \in C^\infty(M)$ such that $e^{2u}g$ is the metric obtained by pulling back \tilde{g} under φ , i.e. $\varphi^*(\tilde{g}) = e^{2u}g$. Pointwise conformal is the special case of conformal equivalence in which one demands that the diffeomorphism φ be the identity map.) This approach has the advantage that, if we seek K as the curvature of a metric \tilde{g} that is pointwise conformal to g , so $\tilde{g} = e^{2u}g$, then one is led to the problem of solving the deceptively innocent-looking nonlinear elliptic equation

$$(1.3) \quad \Delta u = k - Ke^{2u} ,$$

where k and Δ are the Gaussian curvature and Laplacian, respectively, in the given metric g . Consequently, the problem of showing that K is the curvature of a metric \tilde{g} conformally equivalent to g is precisely that of finding a diffeomorphism φ of M such that one can solve

$$(1.4) \quad \Delta u = k - (K \circ \varphi)e^{2u} ,$$

since then the metric $\tilde{g} = (\varphi^{-1})^*(e^{2u}g)$ will have curvature K .

The equation (1.3) may be computed simply as follows. Let $\{\omega_1, \omega_2\}$ be

a local oriented orthonormal coframe field on M for the metric g . If we set $\tilde{\omega}_i = e^u \omega_i$, then $\{\tilde{\omega}_1, \tilde{\omega}_2\}$ is a local oriented orthonormal coframe field for \tilde{g} . Now the Gaussian curvature k of the metric g is determined by the equation $k\omega_1 \wedge \omega_2 = d\varphi_{12}$, where the Riemannian connection form φ_{12} is uniquely determined by the requirements that $\varphi_{12} = -\varphi_{21}$, $d\omega_1 = -\varphi_{12} \wedge \omega_2$, and $d\omega_2 = -\varphi_{21} \wedge \omega_1$. We compute $\tilde{\varphi}_{12}$. Let $du = u_1\omega_1 + u_2\omega_2$. Then

$$d\tilde{\omega}_1 = e^u(du \wedge \omega_1 - \varphi_{12} \wedge \omega_2) = -(u_2\omega_1 - u_1\omega_2 + \varphi_{12}) \wedge \tilde{\omega}_2.$$

Thus $\tilde{\varphi}_{12} = u_2\omega_1 - u_1\omega_2 + \varphi_{12} = \varphi_{12} - *du$, where $*$ is the Hodge star operator. Therefore,

$$K\tilde{\omega}_1 \wedge \tilde{\omega}_2 = d\tilde{\varphi}_{12} = d\varphi_{12} - d*du = k\omega_1 \wedge \omega_2 - \Delta u \omega_1 \wedge \omega_2,$$

from which (1.3) follows immediately.

To summarize, we have asked

QUESTION 2. *If g is a prescribed metric on M and if K satisfies the sign conditions (1.2), is K the curvature of some metric \tilde{g} that is pointwise conformal to g , that is, can one solve (1.3)?*

and

QUESTION 3. *If K and g are as in Question 2, is K the curvature of some metric \tilde{g} conformally equivalent to g , that is, can one find a diffeomorphism φ so that one can solve (1.4)?*

One should observe that a “yes” answer to Question 2 implies a “yes” for 3, and that a “yes” for 3 implies a “yes” answer for Question 1. Of course, it is a priori possible that for a given K satisfying (1.2) the answer to Question 2 is “no” while the answer to 3, and hence 1, is “yes”.

Question 2 has also been posed by L. Nirenberg in a special case. He asked, “Is any given strictly positive function K on S^2 the Gaussian curvature of some metric that is pointwise conformal to the standard metric?” This asks for the solution of (1.3) on S^2 with the standard metric (so $k \equiv 1$) assuming that $K > 0$.

The heart of our results on all of these Questions consists of new existence and non-existence theorems for (1.3). We briefly summarize the current status of these questions considering separately the cases S^2 , P^2 , and $\chi(M) \leq 0$.

S^2 : We began our study of curvature functions with this case. Unfortunately, S^2 remains the only 2-manifold for which our questions are as yet largely unanswered.* One should observe that Questions 1 and 3 are equivalent for S^2 by the Uniformization Theorem, which asserts that any two

* See “added in proof” at the end of the paper.

Riemannian metrics on S^2 are conformally equivalent.

The first progress on S^2 was made by H. Gluck [10] who showed that the answer to Question 1 (and hence to Question 3) is “yes” if one assumes that $K > 0$. Gluck used a clever topological argument to prove that, on the standard $S^2 \subset R^3$, given a smooth function f there is a diffeomorphism φ of S^2 such that

$$(1.5) \quad \int_{S^2} (f \circ \varphi) \vec{n} \, dA = 0 ,$$

where \vec{n} is the unit normal vector field. Applied to $f = 1/K$, condition (1.5) is precisely the integrability condition of the Minkowski problem, the solution [25], [26] of which gives the existence of a closed convex surface in R^3 whose curvature, as a function of the unit normal, is $K \circ \varphi$. This establishes the existence of a metric on S^2 with curvature $K \circ \varphi$; pulling back this metric by φ^{-1} one obtains the desired metric with curvature K . Unfortunately, the method clearly demands the positivity of K .

The answer to Question 2 (and hence 3 and 1) on S^2 is “yes” under the additional hypothesis that $K(x) = K(-x)$. This was proved for K sufficiently close to 1 by D. Koutroufiotis [19] while the general case was established by J. Moser [23] whom we are delighted to have this opportunity to thank for several stimulating conversations. In §7 we indicate briefly how Moser obtained this result from his work [22], [23] on a sharp version of the Trudinger inequality. The general answer to Question 2 on S^2 is however “no”. In fact, in Theorem 8.8 we shall exhibit strictly positive functions K , which are known by Gluck’s work to be curvatures, but which cannot be realized pointwise conformal to the standard metric. Therefore the answer to Nirenberg’s question is also “no”.

Questions 1 and 3 for S^2 as well as sufficient conditions (beyond $K(x) = K(-x)$ with K positive somewhere) for the solvability of (1.3) are open problems.

P^2 : The answer to Question 2 (and hence 3 and 1) is “yes” for the real projective plane P^2 . This follows from the above mentioned results of J. Moser on antipodally symmetric functions on S^2 . It should be noted that P^2 is the *only* 2-manifold for which the answer to Question 2 is “yes”.

$\chi(M) \leq 0$: For these compact 2-manifolds the answers to Questions 3 and 1 are both “yes”. These results are contained in our Theorems 6.2 and 6.3 for the case $\chi(M) = 0$ and in Theorems 11.6 and 11.8 for the case $\chi(M) < 0$. The answer to Question 2 is “no”. Necessary and sufficient conditions (in addition to (1.2)) for the solvability of (1.3) are contained in Theorems 6.1

and 11.1. We wish to thank Melvyn Berger for pointing out to us his application [5] of the calculus of variations to the equation $\Delta u = k - ce^{2u}$ to prove the Uniformization Theorem in the case $\chi(M) \leq 0$. We described our curvature problem and our conjectures in the case of S^2 to Berger; subsequently he was able to apply the variational techniques of [5] to answer Question 2 (and hence 3 and 1) affirmatively for the special case of strictly negative K on manifolds satisfying $\chi(M) < 0$. He also provided a partial solution to Question 2 in the case $\chi(M) = 0$, and showed that if $\chi(M) > 0$, the solution to Question 2 can be reduced to finding a sufficiently sharp version of the Trudinger inequality, see [6] and our equation (7.7). We should also note that in 1898 Poincaré, using non-variational techniques, apparently solved (1.3) on a compact 2-manifold assuming only $\chi(M) < 0$ and $K < 0$, thus providing an affirmative answer to our Question 2 in this case [28, esp. pp. 571–583].

One consequence of our affirmative answers to Question 3 is that *if M is orientable, its complex structures are not distinguished by their curvature functions*. To be specific, if one fixes a complex structure on M , then any curvature function on M is in fact the curvature function of a metric having the given complex structure on M , i.e. all possible curvature functions arise in each complex structure.

Although throughout this paper we will assume that all data (M , metrics g , and curvature K) are smooth, this is merely for convenience. Our proofs go through with little or no change if one makes minimal smoothness hypotheses. For example, without changing any proofs we need only assume that the curvature candidate K is Hölder continuous; in this case the resulting metric with curvature K has Hölder continuous second derivatives.

Before proceeding further we should remark that, on a 2-manifold, one can also consider the related problem of describing the set of curvature forms $\Omega = K\omega$, where ω is the “volume” form. It is an elementary consequence of linear elliptic theory that any 2-form on a compact, connected, oriented 2-manifold M whose integral over M is $2\pi\chi(M)$ is the curvature form of some Riemannian metric on M . In fact Ω can be achieved by a pointwise conformal change of any given metric [35].

The next section contains a summary of our existence and non-existence theorems for (1.3), while subsequent sections contain detailed statements and proofs. We wish to call attention to §4, which contains a rather general asymptotic result that may be of independent interest: in particular, we show that if $u(x; c)$ is a solution of

$$\Delta u + cu = f$$

on a compact manifold, then

$$\lim_{c \rightarrow -\infty} cu(x, c) = f(x) .$$

2. The equation $\Delta u = c - he^u$

By a change of variables one can reduce

$$(2.1) \quad \Delta u = k - Ke^{2u}$$

to a more convenient form. Let v be a solution of $\Delta v = k - \bar{k}$, where $\bar{k} = \int k dA/\text{area}(M)$ is the average of k , and let $w = 2(u - v)$. Then w satisfies

$$(2.2) \quad \Delta w = 2\bar{k} - (2Ke^{2v})e^w .$$

It turns out that equation (2.2) is easier to analyze if one frees it from the geometric situation and instead considers the equation

$$(2.3) \quad \Delta u = c - he^u ,$$

where c is a constant, and h is some prescribed function, with neither c nor h tied to geometric considerations. We shall also let M be any compact, connected manifold of unspecified dimension. In view of our application of (2.1) to open manifolds in [16], [17], in which case K could “blow up” at the boundary, we shall occasionally only assume that $h \in L_p(M)$ for some $p > \dim M$. Although this weak assumption on h slightly complicates some proofs, it avoids an annoying awkward situation in [16], [17] of having to describe how to extend the existence proof here from, say, $h \in C^\infty$, to $h \in L_p$.

It is fascinating to see how the theory of (2.3) changes, depending on the sign of c . Let us summarize this. For this purpose, think of M and $h \in C^\infty$ as being fixed and note the obvious necessary condition found by integrating (2.3) over M :

$$\int_M he^u dV = c \text{Vol}(M) ,$$

which imposes a sign condition on h identical to that on K in (1.2), with c replacing $\chi(M)$. We have found that the existence theory depends much more strongly on c than this simple sign condition. A sketch of the c axis is helpful to understand the results.



$c < 0$: For this range of c , we make no assumption on $\dim M$. A necessary condition for a solution is that $\bar{h} < 0$, in which case there is a critical strictly negative constant $c_-(h)$ such that (2.3) is solvable if $c_-(h) < c < 0$,

but not solvable if $c < c_-(h)$. Estimates for $c_-(h)$ (proved in Section 10) show, in particular, that

- (a) $\bar{h} < 0$ is not a sufficient condition for the solvability of (2.3),
- (b) for fixed $c < 0$, (2.3) is solvable whenever h is “not too positive too often”, and
- (c) $c_-(h) = -\infty$ if and only if $h \leq 0$ ($\neq 0$).

A proof of these assertions is in § 10, where we use the method of upper and lower solutions to obtain existence. The method of upper and lower solutions used in the proofs was learned from [30], [31], although the method is old [20, Ch. 4].

$c = 0$: Here we assume $\dim M \leq 2$. Then, excluding the trivial case $h \equiv 0$, a solution of (2.3) exists if and only if both $\bar{h} < 0$ and h is positive somewhere. The existence is proved using the calculus of variations in § 5. Because of compactness difficulties involving the Sobolev imbedding theorems, we do not know if the sufficiency extends to $\dim M \geq 3$, although the necessary conditions still hold in this case.

$c > 0$: This is the most subtle case. Our information is quite fragmentary. If $\dim M = 1$, so that $M = S^1$, then a solution of (2.3) exists if and only if h is positive somewhere. Here one has an ordinary differential equation to which the calculus of variations applies as in Theorem 7.2. (In the notation there, one shows $c_+(h) = +\infty$ by showing that β can be chosen arbitrarily large.)

If $\dim M = 2$, then there is a constant $0 < c_+(h) \leq \infty$, possibly depending on M , such that a solution exists if h is positive somewhere and if $0 < c < c_+(h)$ (Theorem 7.2). The only information known on $c_+(h)$ is in the special case of the sphere S^2 and the projective plane P^2 , which, for simplicity, we assume have the standard metric of constant curvature 1. For S^2 , Moser [22], [23] proved that if h is positive somewhere, then $c_+(h) \geq 2$, while we can show (§ 8) that for certain positive functions h , $c_+(h) \leq 2$. Thus $c_+(h) = 2$ is the best possible constant that works uniformly for all h which are positive somewhere. In particular, we show that (2.3) is not solvable for any $c \geq 2$ if h is a first order spherical harmonic. On the other hand, as we will show in a future publication, there are other functions for which we can prove non-existence if $c = 2$, but for which we can prove existence for certain $c > 2$. Thus the situation for positive c contrasts markedly with that for negative c where $c_-(h)$ is an absolute cut-off for existence. For P^2 , Moser has shown that if h is positive somewhere, then $c_+(h) \geq 4$. We do not know if this is the best possible constant.

In our work on the cases $c < 0$ and $c = 0$, we found it particularly useful to look at the first order ordinary differential equation

$$u' = c - he^u$$

on $M = S^1$, where one can explicitly find solutions (cf. Chapter VI, B). It is a bit surprising that many properties of (2.3) are already exhibited in this simple example.

The diverse phenomena found here concerning (2.3) indicate some difficulties that will have to be clarified in any general theory for, say,

$$(2.4) \quad \Delta u = f(x, u) .$$

We expect that the theory of (2.3) will serve as a guide and useful example. In [18] we show how some of the methods of this paper apply to (2.4) and other nonlinear elliptic problems. These results are new even for the ordinary differential equation case.

II. Preliminaries

3.* Notation, some inequalities, and linear elliptic equations

Throughout this paper, M will denote a compact, connected (not necessarily orientable) differentiable manifold of unspecified dimension. If M is endowed with a smooth Riemannian metric, then ∇u and Δu will denote, respectively, the gradient and Laplacian of u , while

$$\bar{f} = \frac{1}{\text{vol}(M)} \int_M f \, dV$$

will denote the average of f with respect to the element of volume dV (or area, if $\dim M = 2$, in which case we will write dA) determined by the metric. Also, $\| \cdot \|_p$ will denote the norm in $L_p(M)$, $\| \cdot \|_\infty$ the uniform norm on M , and $\| \cdot \|_{s,p}$ will denote the norm in the Sobolev space $H_{s,p}(M)$ of functions on M whose derivatives up to order s are all in $L_p(M)$. We will write $\| \nabla u \|_p$ instead of the more cumbersome $\| | \nabla u | \|_p$.

Inequalities. We shall need modified versions of standard results. Proofs are sketched for the convenience of geometers who may not be intimate with the techniques of differential equations.

If $\dim M = 2$ and $u \in C^\infty(M)$ with $\bar{u} = 0$, then for any $p \geq 1$ there is a constant c_1 independent of p and u such that

$$(3.1) \quad \| u \|_p \leq c_1 p^{1/2} \| \nabla u \|_2 .$$

The point here is the sharp control of the dependence of the right side on p .

* The reader may wish to skip this section and only refer to it as the need arises.

One first proves this Sobolev inequality for functions $v \in C^\infty(\mathbb{R}^2)$ with compact support in the unit disc $|x| < 1$, i.e. $v \in C_0^\infty(|x| < 1)$. Following [33, p. 125, 128]—with some extra book keeping—we begin with the elementary identity

$$(*) \quad v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla v(x - y) \cdot \frac{y}{|y|^2} dy ,$$

where dy is the element of area. Since v has support in $|x| < 1$, the integration is performed only over $|y| < 2$, so replace $y/|y|^2$ by $\varphi(y) = y/|y|^2$ for $|y| \leq 2$ and let $\varphi(y) = 0$ elsewhere. Then by Young’s inequality [36, p. 37]

$$\|v\|_p \leq \frac{1}{2\pi} \|\varphi\|_s \|\nabla v\|_2 , \quad \text{where } \frac{1}{p} = \frac{1}{s} - \frac{1}{2}, \quad s \geq 1 .$$

But if $s < 2$, then by an explicit computation

$$\begin{aligned} \|\varphi\|_s &= \left(2\pi \frac{2^{2-s}}{2-s}\right)^{1/s} \leq c(2-s)^{-1/s} \\ &= c\left(\frac{p}{2s}\right)^{1/p+1/2} \leq cp^{1/2}p^{1/p}(2s)^{-1/2} \leq c'p^{1/2} , \end{aligned}$$

where the constants c and c' are independent of p . This proves (3.1) for $v \in C_0^\infty(|x| < 1)$.

A partition of unity argument combined with the above inequality for $v \in C_0^\infty(|x| < 1)$ shows that there is a constant c_2 such that for any $u \in C^\infty(M)$

$$(3.1)' \quad \|u\|_p \leq c_2 p^{1/2} (\|u\|_2 + \|\nabla u\|_2) .$$

Inequality (3.1) now follows from (3.1)' and the Poincaré inequality, which states that there is a constant c_3 such that if $u \in C^\infty(M)$ with $\bar{u} = 0$, then

$$(3.2) \quad \|u\|_2 \leq c_3 \|\nabla u\|_2 .$$

(The shortest proof of (3.2) is from the variational characterization of the first non-zero eigenvalue of the Laplacian on M .)

Another immediate consequence of the Poincaré inequality (3.2) is that there is a constant c_4 such that for any $u \in C^\infty(M)$ with $\bar{u} = 0$, one has

$$(3.3) \quad \|u\|_{1,2} \leq c_4 \|\nabla u\|_2 .$$

Inequality (3.1) is needed to extend an inequality of Trudinger [34] from \mathbb{R}^2 to compact manifolds (see also [4] for a much more complicated proof). The extended inequality asserts that if $\dim M = 2$, then there are positive constants β, γ such that for any $u \in H_{1,2}(M)$ with $\bar{u} = 0$ and $\|\nabla u\|_2 \leq 1$, one has

$$(3.4) \quad \int_M e^{\beta u^2} dA \leq \gamma .$$

To prove (3.4), we follow Trudinger and use the power series for exp, estimating the individual terms by (3.1), to find that

$$\int_M e^{\beta u^2} dA = \sum \frac{\beta^k}{k!} \|u\|_{2k}^{2k} \leq \sum \frac{(2c_1^2 \beta k)^k}{k!},$$

which converges to some number γ if β satisfies $2c_1^2 \beta e < 1$.

Moser [22] has given an independent and quite different proof of (3.4) for the special cases $M = S^2$ and $M = P^2$. He has shown that in both of these cases the best constant $\beta = 4\pi$.

We shall need two consequences of (3.4). The first is that if $\dim M = 2$, there exist positive constants β, γ such that for any $u \in H_{1,2}(M)$ and any constant $\alpha > 0$,

$$(3.5) \quad \int_M e^{\alpha|u|} dV \leq \gamma \exp \left[\alpha |\bar{u}| + \frac{(\alpha \|\nabla u\|_2)^2}{4\beta} \right].$$

Consequently (for $\dim M = 2$),

$$(3.6) \quad \text{if } u \in H_{1,2}(M), \text{ then } e^u \in L_p(M), \quad \text{for all } 1 \leq p < \infty.$$

To prove (3.5), let $a = \|\nabla u\|_2$ and define v by $u = \bar{u} + av$. Then $\|\nabla v\|_2 = 1, \bar{v} = 0$, and $\alpha a|v| \leq \beta v^2 + (\alpha a)^2/(4\beta)$, where β is the constant in (3.4). Now apply (3.4) to complete the proof of (3.5).

The second consequence of (3.4) is

$$(3.7) \quad \text{Assume } \dim M = 2. \text{ If } u_j \in H_{1,2}(M) \text{ and } u_j \rightarrow u \text{ weakly in } H_{1,2}(M), \text{ then } \exp u_j \rightarrow \exp u \text{ strongly in } L_2(M).$$

In order to prove (3.7), we use the mean value theorem to see that $|e^t - 1| \leq |t|e^{|t|}$, and the Rellich-Kondrashov compactness theorem which asserts that if $u_j \rightarrow u$ weakly in $H_{1,2}(M)$ then $u_j \rightarrow u$ strongly in $L_4(M)$. Combining these facts and the Schwarz inequality several times we find that

$$\begin{aligned} \int_M |e^{u_j} - e^u|^2 dV &= \int_M e^{2u} |e^{u_j - u} - 1|^2 dV \\ &\leq \int_M e^{2u} e^{2|u_j - u|} |u_j - u|^2 dV \\ &\leq \left(\int_M e^{8u} dV \right)^{1/4} \left(\int_M e^{8|u_j - u|} dV \right)^{1/4} \|u_j - u\|_4^2. \end{aligned}$$

Now (3.5) shows that the first two terms on the right are bounded, while we picked u_j so that $\|u_j - u\|_4^2 \rightarrow 0$.

Since (3.1) fails for $p = \infty$ or $\dim M \geq 3$, we shall need another Sobolev inequality which asserts that if $p > \dim M$, there is a constant $c_s > 0$ such that for any $u \in H_{1,p}(M)$

$$(3.8) \quad \|u\|_\infty \leq c_s \|u\|_{1,p},$$

and the obvious extension to bound $\|\nabla u\|_\infty$ if $u \in H_{2,p}(M)$ ((3.8) is proved locally in \mathbf{R}^n by Hölder's inequality applied to the \mathbf{R}^n version of the identity (*) after (3.1), and extended globally to M by a partition of unity). A consequence of (3.8) which we shall frequently use is that if $u \in H_{2,p}(M)$ for some $p > \dim M$, then u and ∇u are continuous.

Linear elliptic equations. Here too we shall need modified versions of standard facts. Again, proofs are only sketched.

If A is a second order elliptic operator on M with smooth coefficients (i.e., $u \in C^\infty(M)$ implies $Au \in C^\infty(M)$), then for any $p > 1$ there is a constant $c_6 > 0$ such that for all $u \in C^\infty(M)$

$$(3.9) \quad \|u\|_{2,p} \leq c_6 (\|Au\|_p + \|u\|_{1,p}) .$$

This is the L_p version of the fundamental elliptic inequality. It is proved using a partition of unity on M , beginning from the corresponding local L_p estimate in \mathbf{R}^n [1, §7], [2, §15].

As a consequence of (3.9) one can show [1, p. 439] that if $u \in L_2(M)$ is a weak solution of $Au = f$ (i.e., in the $L_2(M)$ inner product $\langle u, A^*\varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C^\infty(M)$, where A^* is the formal adjoint of A), and if $f \in L_p(U)$ for some $2 \leq p < \infty$ and some open $U \subset M$, then $u \in H_{2,p}(U)$.

We shall be combining the L_p regularity just stated with the regularity theorems that follow from the Schauder estimates. If $0 < \alpha \leq 1$ and $j \geq 0$ is an integer, then $C^{j+\alpha}(M)$ denotes the space of functions whose j^{th} derivatives satisfy a Hölder condition with exponent α . The Schauder regularity theory asserts that if $Au \in C^{j+\alpha}(U)$ for some open $U \subset M$, then $u \in C^{j+2+\alpha}(U)$ [9, p. 339, 345], [7, pp. 240-242].

In §9, for use in [16] we will be considering the operator

$$(3.10) \quad Lu = \Delta u - ku ,$$

where $k \in L_p(M)$ for some $p > \dim M$ and $k \geq \text{const} > 0$. Thus we need extensions of several results to this case of unbounded coefficients. First, there are constants c_7 and γ such that for any $u \in H_{2,p}(M)$,

$$(3.11) \quad \|u\|_{2,p} \leq c_7 \|Lu\|_p ,$$

$$(3.12) \quad \|u\|_\infty + \|\nabla u\|_\infty \leq \gamma \|Lu\|_p .$$

One proves (3.11) using the triangle inequality and (3.9) with $A = \Delta$, to conclude that (3.9) holds with $A = L$. But $k \geq \text{const} > 0$ implies that $\ker L = 0$, so (3.11) follows from (3.9) with $A = L$ by a standard argument, cf. [2, §7, Remark 2]. (3.12) is a consequence of (3.11) and (3.8).

(3.13) *Existence for (3.10): $L: H_{2,p}(M) \rightarrow L_p(M)$ is a bijection.*

That $\ker L = 0$ was mentioned just above. To prove existence, one applies (3.11) to prove easily that if $a \in C^\infty(M)$ satisfies $\|a - k\|_p < 1/(2c_3c_7)$, then for any $u \in C^\infty(M)$

$$(3.14) \quad \|u\|_{2,p} \leq 2c_7 \|\Delta u - au\|_p.$$

Now if $f \in L_p(M)$ is prescribed, one chooses $a_j, f_j \in C^\infty(M)$, where $a_j \geq \text{const} > 0$, such that $a_j \rightarrow k$ and $f_j \rightarrow f$ in $L_p(M)$. By standard elliptic theory, one can uniquely solve $\Delta u_j - a_j u_j = f_j$, giving $u_j \in C^\infty(M)$. Inequality (3.14) shows that the u_j 's converge in $H_{2,p}(M)$ to the desired solution $u \in H_{2,p}(M)$ of $Lu = f$.

(3.15) *Maximum principle for (3.10): if $u \in H_{2,p}(M)$ satisfies $Lu \geq 0$, then $u \leq 0$.*

We adapt the method of Stampacchia [32, p. 387]. Let $w(x) = \max\{0, u(x)\}$. Since $u \in H_{2,p}(M)$ for $p > \dim M$, then by (3.8) $u \in C^1(M)$. Thus $w \in H_{1,2}(M)$. But

$$0 \leq \int_M wLu \, dV = - \int_M (|\nabla w|^2 + kw^2) \, dV.$$

Since $k \geq \text{const} > 0$, this implies $w \equiv 0$. Therefore $u \leq 0$.

$$4.* \quad \Delta u + cu = f \text{ as } c \rightarrow -\infty$$

In the course of our work, we will need to know the behavior of the solution $u(x; c)$ of $\Delta u + cu = f$ (on compact M) as $c \rightarrow -\infty$. The result is, if f is sufficiently smooth, then $cu(x; c) \rightarrow f(x)$ uniformly for $x \in M$. We are somewhat surprised that this ‘‘classical sounding’’ fact does not seem to have been observed previously. Our proof readily generalizes to many other situations.

The key is the following functional analysis lemma. Here B_1 and B_2 are Banach spaces and B_2 a subspace of B_1 such that the natural injection $B_2 \rightarrow B_1$ is continuous, $\|\cdot\|_1$ is the norm in B_1 , and $L : B_2 \rightarrow B_1$ is a continuous linear map.

ASYMPTOTIC LEMMA 4.1. *Assume that $(L + \alpha I) : B_2 \rightarrow B_1$ is invertible for all $\alpha \leq 0$ and that*

$$(4.2) \quad \|(L + \alpha I)^{-1}\| \equiv \sup_{Z \in B_1} \frac{\|(L + \alpha I)^{-1}Z\|_1}{\|Z\|_1} \leq m(\alpha)$$

where $m(\alpha) \rightarrow 0$ as $\alpha \rightarrow -\infty$. If $Y \in B_2$ (not just B_1), let X_α be the unique solution of $LX_\alpha + \alpha X_\alpha = Y$. Then

$$(4.3) \quad \lim_{\alpha \rightarrow -\infty} \|\alpha X_\alpha - Y\|_1 = 0,$$

* The reader may skip this section and refer to it when needed in §10.

that is, $\alpha X_\alpha \rightarrow Y$.

Proof. Let $G = L^{-1}$, so $G : B_1 \rightarrow B_2$. Also note that for any $\alpha \leq 0$, $(I + \alpha G) : B_2 \rightarrow B_2$ is invertible; in fact, $(I + \alpha G)^{-1} = (L + \alpha I)^{-1}L$. Therefore, for any $Y \in B_2$,

$$\|(I + \alpha G)^{-1}Y\|_1 \leq \|(L + \alpha I)^{-1}\| \|LY\|_1 \leq m(\alpha) \|LY\|_1 .$$

Now $LX_\alpha + \alpha X_\alpha = Y$ implies that

$$- \alpha X_\alpha + Y = (I + \alpha G)^{-1}Y .$$

Thus as $\alpha \rightarrow -\infty$,

$$\|\alpha X_\alpha - Y\|_1 = \|(I + \alpha G)^{-1}Y\|_1 \leq m(\alpha) \|LY\|_1 \rightarrow 0 . \quad \text{Q.E.D.}$$

ASYMPTOTIC THEOREM 4.4. *For $c < 0$, let $u(x; c)$ denote the unique solution of $\Delta u + cu = f \in C^\infty(M)$ on a compact manifold M . Then*

$$\lim_{c \rightarrow -\infty} cu(x; c) = f(x) ,$$

where the convergence is uniform on M .

Proof. We apply the Asymptotic Lemma with $B_1 = H_{s,2}(M)$ and $B_2 = H_{s+2,2}(M)$, for s a positive even integer, $s = 2k$. Let $L = \Delta - I$. Then $L + \alpha I : H_{s+2,2} \rightarrow H_{s,2}$ is continuous and has a continuous inverse for any $\alpha \leq 0$ by standard elliptic theory. If we let $\psi = (L + \alpha I)^{-1}\varphi$ for some $\varphi \in H_{s,2}$, then (4.2) will be established if we can show that there is a function $m(\alpha) \rightarrow 0$ as $\alpha \rightarrow -\infty$ such that

$$\|\psi\|_{s,2} \leq m(\alpha) \|(L + \alpha I)\psi\|_{s,2}$$

for all $\psi \in H_{s+2,2}$. Now the fundamental inequality for elliptic operators (cf. (3.11)) shows that the $H_{s,2}$ norm of φ is equivalent to the L_2 norm of $L^k\psi = (\Delta - I)^k\psi$ (recall that $s = 2k$). Thus we can consider the $H_{s,2}$ inner product and norm to be defined by

$$\langle \psi, \varphi \rangle_{s,2} = \langle L^k\psi, L^k\varphi \rangle ,$$

where \langle , \rangle is the $L_2(M)$ inner product. Consequently,

$$\begin{aligned} - \langle L\psi, \psi \rangle_{s,2} &= \langle \psi, \psi \rangle_{s,2} - \langle \Delta L^k\psi, L^k\psi \rangle \\ &= \|\psi\|_{s,2}^2 + \|\nabla L^k\psi\|_2^2 \geq 0 , \end{aligned}$$

where we have integrated by parts to obtain the second line. Therefore, for $\alpha < 0$

$$- \alpha \|\psi\|_{s,2}^2 \leq - \langle L\psi + \alpha\psi, \psi \rangle_{s,2} \leq \|(L + \alpha I)\psi\|_{s,2} \|\psi\|_{s,2} .$$

This verifies (4.2) with $m(\alpha) = 1/|\alpha|$, that is,

$$\|\psi\|_{s,2} \leq \frac{1}{|\alpha|} \|(L + \alpha I)\psi\|_{s,2} .$$

If we let $\alpha = c + 1$, then $f = \Delta u + cu = Lu + \alpha u$. Thus by the Asymptotic Lemma,

$$\lim_{\alpha \rightarrow -\infty} \|\alpha u - f\|_{s,2} = 0$$

for any positive integer s . By the Sobolev imbedding theorem [7, Lemma 3, p.194], if s is sufficiently large, we conclude the convergence is uniform on M ,

$$\lim_{c \rightarrow -\infty} (c + 1)u(x; c) = f(x) .$$

This completes the proof.

III. The case $c = 0$

5. Analysis of $\Delta u = -he^u$

Throughout this section we shall assume that the compact connected manifold M is of dimension 2 and has a given Riemannian structure. No assumption is made on the Euler characteristic. We shall consider the equation

$$(5.1) \quad \Delta u = -he^u ,$$

where h is a smooth function on M , and we exclude the trivial case $h \equiv 0$ in which case the solutions of (5.1) are precisely the constant functions on M .

By integrating both sides of (5.1) over M one obtains the "linear theory" necessary condition that $\int_M he^u dA = 0$. For smooth h (not $\equiv 0$) this condition requires that h change sign. The key to the theory of (5.1) (and the missing ingredient in the treatment in [6]) is the observation of an additional necessary condition which reflects the particular nonlinear nature of this equation, namely that $\int_M h dA < 0$. To obtain this, observe that $u \equiv \text{const}$ cannot be a solution, multiply (5.1) by e^{-u} , and integrate over M to find

$$(5.2) \quad \int_M h dA = - \int_M e^{-u} \Delta u dA = - \int_M e^{-u} |\nabla u|^2 dA < 0 .$$

Here we have used an integration by parts (that is, an application of the divergence theorem which, one should observe, is perfectly valid for non-orientable Riemannian manifolds [11, p. 388]). We shall now use the direct method of the calculus of variations to show that these two necessary conditions are also sufficient for the existence of a solution to (5.1).

THEOREM 5.3. *Let h (not $\equiv 0$) belong to $C^\infty(M)$. Then the conditions*

- (i) h changes sign,
- (ii) $\int_M h \, dA < 0$,

are necessary and sufficient for the existence of a solution $u \in C^\infty(M)$ for equation (5.1).

Proof. We have just observed the necessity. For the sufficiency we use the calculus of variations.

Define a set of functions B by

$$B = \left\{ v \in H_{1,2}(M) : \int_M h e^v \, dA = 0, \bar{v} = 0 \right\}.$$

Since h changes sign, it is easy to see that B is not empty. We shall minimize the functional

$$(5.4) \quad J(v) = \int_M |\nabla v|^2 \, dA, \quad \text{for } v \in B.$$

Clearly $J \geq 0$. Let $a = \inf J(v)$ for $v \in B$. Say $\{v_n\} \in B$ is a minimizing sequence, so $J(v_n) \downarrow a$. Because B is not empty, there is some $v_0 \in B$. Let $b = J(v_0)$. Then we can assume $J(v_n) \leq b$. It follows from (3.3) that $\|v_n\|_{1,2}^2 \leq \text{const } J(v_n) \leq \text{const}$ for all n . Because the unit ball in any Hilbert space is weakly compact, we conclude that there is some $v \in H_{1,2}(M)$ such that a subsequence of the v_n 's, which we relabel v_n , converges weakly to v . This implies that $\bar{v} = 0$. Since, by (3.7), e^{v_n} converges to e^v in L_2 , we obtain $\int_M h e^v \, dA = 0$. Therefore $v \in B$.

To conclude that v minimizes J for all $v \in B$, we use the general result that whenever v_n converges to v weakly in a Hilbert space, then $\|v\| \leq \liminf \|v_n\|$ (proof: let $z = v/\|v\|$ and note that $\|v_n\| \geq \langle z, v_n \rangle \rightarrow \|v\|$). In our case, the Hilbert space is the subspace of $H_{1,2}(M)$ with $\bar{v} = 0$. Because of inequality (3.3), $\sqrt{J(v)}$ is a norm equivalent to the norm $\|\cdot\|_{1,2}$ on this subspace. Thus $J(v) \leq J(v_n)$ for all n . Therefore v minimizes J in B .

Since v minimizes J in B , by standard Lagrange multiplier theory we find that there are constants λ and μ such that for any $\varphi \in H_{1,2}(M)$

$$(5.5) \quad \int_M [2\nabla v \cdot \nabla \varphi + \lambda h e^v \varphi + \mu \varphi] \, dA = 0.$$

This is the Euler-Lagrange equation. Integrating by parts, taking a derivative off of v , one immediately sees that if we let $F \equiv (\lambda h e^v + \mu)/2$, then v is a weak solution of $\Delta u = F$. Since $v \in H_{1,2}$, we find from (3.6) that $e^v \in L_p(M)$ for all $p \geq 1$, in particular for some $p > 2 = \dim M$ which we now fix. Since F then belongs to $L_p(M)$, it follows from the L_p regularity theory (§3

for weak solutions of linear elliptic equations that v lies in $H_{2,p}$ and hence in C^1 by (3.8). Then F lies in C^1 , so by the Schauder estimates (§3) $v \in C^2$. Continuing inductively one finds that $v \in C^\infty$. It remains to evaluate the Lagrange multipliers λ and μ . The special case $\varphi \equiv 1$ in (5.5) gives $\mu = 0$; and the special case $\varphi = e^{-v}$, in view of the assumption $\int_M h \, dA < 0$, shows that $\lambda < 0$. Thus we can write $-\lambda = 2e^\gamma$ for some constant γ . Then $u = v + \gamma$ is the desired solution $u \in C^\infty(M)$ of $\Delta u = -he^u$. Q.E.D.

6. Curvatures of compact 2-manifolds with $\chi(M) = 0$

As an immediate consequence of the previous section, we can answer the three Questions posed in §1 for compact 2-manifolds with zero Euler characteristic, $\chi(M) = 0$, i.e., for the torus and Klein bottle. We first consider Question 2.

THEOREM 6.1. *Let M be a compact 2-manifold with $\chi(M) = 0$ and let g be a given metric on M . Then, $K \in C^\infty(M)$ is the curvature of a metric \tilde{g} that is pointwise conformal to g if and only if either $K \equiv 0$ or both*

(i) $\int_M Ke^{2v} \, dA < 0$, where $\Delta v = k$ with k the curvature of g ,

and

(ii) K changes sign.

Remark. The inequality (i) should not be confused with the Gauss-Bonnet condition $\int Ke^{2u} \, dA = 0$. For example, if one begins with the standard flat metric having $k \equiv 0$ on the torus, then $v \equiv \text{const}$ in (i), so the inequality reads $\bar{K} < 0$.

Proof. Given the metric g , then by the Gauss-Bonnet theorem its curvature k satisfies $\bar{k} = 0$ (which also shows that one can solve $\Delta v = k$). Thus by the change of variables (2.1)–(2.2), we seek a solution of

$$\Delta w = -2Ke^{2v}e^w,$$

which is precisely (5.1) with $h = 2Ke^{2v}$. The conclusion now follows from Theorem 5.3. Q.E.D.

Next we answer Question 3.

THEOREM 6.2. *Let M be a compact 2-manifold with $\chi(M) = 0$ and let g be a given metric on M . Then a given function $K \in C^\infty(M)$ is the curvature of a metric \tilde{g} that is conformally equivalent to g if and only if either K changes sign or else $K \equiv 0$.*

Proof. Necessity that K change sign if it is not $\equiv 0$ follows from (1.2).

If $K \equiv 0$, then K is trivially the curvature of a metric pointwise conformal to g . For sufficiency in the case that $K \not\equiv 0$, we must find a diffeomorphism φ of M such that one can solve (1.4). In view of Theorem 6.1, we must merely find φ such that

$$\int_M (K \circ \varphi) e^{2v} dA < 0,$$

which is clearly possible since K is negative in some open set. Q.E.D.

As an immediate consequence of Theorem 6.2 we can resolve Question 1 for this case.

THEOREM 6.3. *Let M be a compact 2-manifold with $\chi(M) = 0$, i.e. the torus or Klein bottle. Then a given function $K \in C^\infty(M)$ is the curvature of some metric on M if and only if either K changes sign or $K \equiv 0$.*

IV. The case $c > 0$

7. Analysis of $\Delta u = c - he^u$ with $c > 0$

We again assume that M is an arbitrary compact connected 2-manifold possessing a given Riemannian structure with area element dA . We consider the equation

$$(7.1) \quad \Delta u = c - he^u$$

with $c > 0$ constant and with h a smooth function on M . If there is a solution u of (7.1) then upon integrating the equation over M one immediately observes that h must be positive somewhere. This condition is, in fact, sufficient for the existence of a solution to (7.1) for all sufficiently small $c > 0$.

THEOREM 7.2. *Consider (7.1) with $c > 0$ and $h \in C^\infty(M)$.*

(a) *A necessary condition for the existence of a solution to (7.1) is that h be positive somewhere on M .*

(b) *If h is positive somewhere, there is a constant $c_+ > 0$ depending on h (which we write $c_+(h)$) such that (7.1) has a solution $u \in C^\infty(M)$ for $0 < c < c_+(h)$. Moreover*

$$(7.3) \quad c_+(h) \geq 2\beta/A$$

where β is the constant in the Trudinger inequality (3.4) and A is the area of M .

The proof of this theorem is essentially contained in [6] and [23] so we just indicate the steps briefly. One solves (7.1) by minimizing the functional

$$(7.4) \quad J(u) = \int_M \left(\frac{1}{2} |\nabla u|^2 + cu \right) dA$$

on the subset B of $H_{1,2}(M)$ defined by the constraint

$$(7.5) \quad \int_M h e^u dA = cA .$$

Observe that the assumption that h is positive somewhere guarantees that the constraint set B is not empty. Writing $u = v + \bar{u}$, so $\bar{v} = 0$, and solving for \bar{u} in (7.5) one sees that J can be expressed as

$$(7.6) \quad J(u) = \int_M \frac{1}{2} |\nabla v|^2 dA - cA \log \int_M h e^v dA + cA \log cA .$$

Combining this with the estimate in (3.5), we find that

$$(7.7) \quad J(u) \geq \frac{1}{4\beta} (2\beta - Ac) \|\nabla v\|_2^2 + \text{const} .$$

Hence J is bounded below if $c \leq 2\beta/A$. In the case $c = 2\beta/A$, the functional J may have no minimum, in fact no critical points whatsoever, for certain functions h which are positive somewhere even though J is bounded below. We shall exhibit cases of this phenomenon in the following section for $M=S^2$. If, however, one has strict inequality, $c < 2\beta/A$, then one can use (7.7) to show that minimizing sequences remain in a fixed ball in $H_{1,2}(M)$ which is weakly compact. Thus one can select a weakly converging subsequence and the standard variational argument (as in Theorem 5.3, for example) yields the existence of a solution $u \in C^\infty(M)$ for (7.1).

It follows from the work [22] of Moser on the best possible value for β on the standard 2-sphere S^2 (with constant curvature 1) and on the standard real projective plane P^2 that $c_+(h) \geq 2$ for all functions h positive somewhere on S^2 and that $c_+(h) \geq 4$ both for antipodally symmetric h which are positive somewhere on S^2 and for functions h positive somewhere on P^2 (see [23]). Since the relevant value of c for the solution of (2.1) on either the standard S^2 or P^2 is 2, Moser concludes that all functions which are antipodally symmetric and positive somewhere on S^2 are Gaussian curvatures of metrics on S^2 and all functions positive somewhere on P^2 are Gaussian curvatures of metrics on P^2 . Moreover, in each of these cases the metric realizing the curvature candidate can be chosen pointwise conformal to the standard metric.

We shall show in the next section that there are functions h , positive somewhere on S^2 , for which (7.1) has no solutions for any $c \geq 2$. Consequently $c_+(h) = 2$ is the largest value of c_+ which works uniformly for all h positive somewhere on S^2 . We shall also show that there are strictly positive h for which there are no solutions of (7.1) for the case $c = 2$.

It is of interest to determine additional information on the dependence of c_+ on h and on M ; and, in general, to determine the structure of the set of positive c 's for which (7.1) has a solution for a given h and M .

8. Integrability conditions on S^2

In this section we derive a new integrability condition for the equation $\Delta u = c - he^u$ with respect to the standard metric on the 2-sphere S^2 for certain positive values of c .

To begin, we observe the following identity which holds for any pair of smooth functions u, F on any Riemannian manifold.

Basic Identity:

$$(8.1) \quad 2\Delta u(\nabla F \cdot \nabla u) = \nabla(2(\nabla F \cdot \nabla u)\nabla u - |\nabla u|^2\nabla F) - (2H_F - (\Delta F)g)(\nabla u, \nabla u).$$

Here g denotes the metric tensor and H_F is the Hessian or 2nd covariant derivative of F . (In Euclidean space, the matrix of the symmetric bilinear form H_F with respect to the canonical bases is just the matrix of 2nd partial derivatives of F .) In the notation of the classical tensor calculus this identity becomes

$$2u^j_{;j}(F^i u_i) = 2((F^i u_i)u^j)_{;j} - ((u^i u_i)F^j)_{;j} - 2F^i_{;j}u_i u^j + F^j_{;j}u^i u_i,$$

a formula whose validity is readily verified.

Now we restrict our attention to S^2 with its standard metric. If F is a 1st order spherical harmonic (i.e. the restriction to S^2 of a linear function in \mathbf{R}^3) then

$$(8.2) \quad \Delta F = -2F \quad \text{and} \quad 2H_F - (\Delta F)g = 0.$$

Thus for such F the basic identity becomes

$$(8.3) \quad \Delta u(\nabla F \cdot \nabla u) \sim 0,$$

where we use \sim to denote equality modulo terms which are divergences. We now apply (8.3) to derive an integrability condition for solutions u of the equation $\Delta u = c - he^u$. Replacing Δu in (8.3) by $c - he^u$ we obtain

$$(8.4) \quad c\nabla u \cdot \nabla F \sim he^u \nabla u \cdot \nabla F.$$

Consider the left hand side of (8.4). Taking a derivative off of F and placing it on ∇u , again substituting $c - he^u$ for Δu , and observing that $c^2 F$ is a divergence since F is a spherical harmonic, we get that

$$(8.5) \quad c\nabla u \cdot \nabla F \sim -cF\Delta u = -cF(c - he^u) \sim chFe^u.$$

On the right hand side of (8.4) we observe that $e^u \nabla u = \nabla e^u$, we remove the derivative from e^u and place it on $h \nabla F$, and we use the fact $\Delta F = -2F$ to obtain

$$(8.6) \quad h e^u \nabla u \cdot \nabla F \sim -e^u \nabla (h \nabla F) = 2h e^u F - e^u \nabla h \cdot \nabla F .$$

Combining (8.5) and (8.6) with (8.4) we see that

$$(8.7) \quad e^u \nabla h \cdot \nabla F \sim (2 - c) h F e^u .$$

Integrating both sides of (8.7) over S^2 proves the following.

THEOREM 8.8. *If u is a solution of the equation*

$$(8.9) \quad \Delta u = c - h e^u$$

on the standard 2-sphere, then

$$(8.10) \quad \int_{S^2} e^u \nabla h \cdot \nabla F \, dA = (2 - c) \int_{S^2} e^u h F \, dA ,$$

for all spherical harmonics F of degree 1.

In the case $c = 2$, we see that (8.9) has no solutions for any function h such that $\nabla h \cdot \nabla F_0$ has a fixed sign for some spherical harmonic F_0 of degree 1, in particular for all functions h of the form $F_0 + \text{const}$. In the case $c > 2$, if $h = F_0$ is a spherical harmonic of degree 1, then (8.9) has no solutions since the two sides of (8.10) will have opposite signs for $F = F_0$.

On the standard 2-sphere, the equation (1.3) becomes $\Delta u = 1 - K e^{2u}$, so it follows from Theorem 8.8 that *a function K for which $\nabla K \cdot \nabla F_0$ has a fixed sign for some spherical harmonic F_0 of degree 1 cannot be realized as the curvature of a metric pointwise conformal to the standard metric on S^2* . In particular, there are strictly positive such functions K , which are known by Gluck’s work [10] to be curvatures, but which cannot be realized pointwise conformal to the standard metric. This shows that the answers to L. Nirenberg’s question and to our Question 2 for S^2 are both “no” for certain K . Note also that the set of K for which one can solve $\Delta u = 1 - K e^{2u}$ is not open, for there exist solutions (indeed a 3-parameter family of them) for $K \equiv 1$, but there are no solutions for $K = 1 + F$ where F is any non-trivial spherical harmonic of degree 1. Questions 1 and 3 remain open for S^2 . We believe that the answers to both will be “yes”.

If one uses spherical coordinates ($z = \cos \varphi$, $x = \sin \varphi \cos \theta$, $y = \sin \varphi \sin \theta$) and considers the special case of (8.10) in which $c = 2$ and $F = \cos \varphi$, then (8.10) becomes

$$(8.11) \quad \int_0^\pi \left\{ \int_0^{2\pi} e^u h_\varphi \sin^2 \varphi \, d\theta \right\} d\varphi = 0 .$$

It was this form of (8.10) to which we were first led by our observation of the non-existence of rotationally symmetric (function of φ alone) solutions for $\Delta u = 2 - he^u$ given rotationally symmetric data h (see [15]). One can directly verify (8.11) by using the equation and several integrations by parts.

We should remark that our proof of Theorem 8.8, beginning with the identity (8.3), has the advantage that it generalizes to give integrability conditions on S^n for the equation describing the change of scalar curvature under pointwise conformal change of metric (see [17]) and also to give Pohožaev's proof of the non-existence of positive solutions in the case of the Dirichlet problem for the equation $\Delta u = -\lambda u^{(n+2)/(n-2)}$, $\lambda > 0$, on a starlike domain in \mathbf{R}^n (see [27] and [18]).

Another perhaps more conceptual method of proving Theorem 8.8 is based on an idea of G. Rosen [29]. A solution u of (8.9) is a critical point of the functional

$$(8.12) \quad J(u) = \int_M \left(\frac{1}{2} |\nabla u|^2 + cu - he^u \right) dA .$$

Let $\varphi_\lambda : S^2 \rightarrow S^2$ be the (conformal) diffeomorphism induced on S^2 under stereographic projection from \mathbf{R}^2 by the map $\Phi_\lambda : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $\Phi_\lambda(p) = \lambda p$. Then $\varphi_1 = \text{id}$, so

$$(8.13) \quad \left. \frac{dJ(u \circ \varphi_\lambda)}{d\lambda} \right|_{\lambda=1} = 0 .$$

If one carries out the computation of (8.13) one obtains (8.10) for the case $F = z|S^2 = \cos \varphi$.

One should observe that the value of c at which (8.10) first yields obstructions to the solvability of (8.9) is 2, which is the 1st non-zero eigenvalue of $-\Delta$ on S_2 . An investigation of the case $c = 6$, which corresponds to the 2nd eigenvalue of $-\Delta$, shows that new phenomena arise here also. Beginning with the basic identity (8.1) and proceeding as above, one shows that if F is a 2nd order spherical harmonic (i.e., satisfies $\Delta F = -6F$), then for any solution u of (8.9) one has

$$(8.14) \quad e^u \nabla h \cdot \nabla F \sim (6 - c) h F e^u - \frac{1}{2} \left(2H_F - (\Delta F) g \right) (\nabla u, \nabla u) .$$

We now use this to show that there are rotationally symmetric functions h , say $h = 3 \cos^2 \varphi - 1$ for which $\Delta u = 6 - he^u$ has no rotationally symmetric solutions. Integrating (8.14) over S^2 with $c = 6$, with $F = 3 \cos^2 \varphi - 1$, and with u a function of φ alone, one obtains

$$(8.15) \quad \int_{S^2} e^u h_\varphi F_\varphi dA = -18 \int_{S^2} (\sin^2 \varphi) u_\varphi^2 dA ,$$

which is clearly not satisfied for any h of the form $a \cos^2 \varphi + b$ where a and b are constants with $a > 0$.

V. The case $c < 0$

9. The method of upper and lower solutions

Here we are trying to solve

$$(9.1) \quad \Delta u = c - he^u$$

on a compact connected manifold M , where $c < 0$ is a constant. If one tries to use the calculus of variations to solve this, then one can prove that a solution exists if $\dim M = 2$ and $h \leq 0$ ($\neq 0$), cf. [6] for the case $h < 0$, which was earlier treated non-variationally in [28, esp. pp. 571-583]. If h is positive on some open set, then one can show that the functional one uses in the variational method is unbounded, even if $\dim M = 1$. Thus we use a different approach to prove existence of solutions to (9.1) for functions h that are occasionally positive.

We shall use the method of upper and lower solutions, which has the additional virtue that it does not presume anything about $\dim M$. For our applications in [16, 17] to the open manifold case, we will assume here that $h \in L_p(M)$ for some $p > \dim M$, although the simpler case $h \in C^\infty(M)$ suffices for this paper. We shall call $u_-, u_+ \in H_{2,p}(M)$ a lower (respectively, upper) solution of (9.1) if

$$(9.2) \quad \Delta u_- - c + he^{u_-} \geq 0, \quad \Delta u_+ - c + he^{u_+} \leq 0$$

on M . Inequalities, of course, hold almost everywhere.

LEMMA 9.3. *Let $c < 0$ and $p > \dim M$ be constants. If there exist upper and lower solutions, $u_+, u_- \in H_{2,p}(M)$ of (9.1) and if $u_- \leq u_+$, then there is a solution $u \in H_{2,p}(M)$ of (9.1). Moreover, $u_- \leq u \leq u_+$, and u is C^∞ in any open set where h is C^∞ .*

Remark. If one assumes $h \in C^\infty(M)$, then one can follow the proof in Courant-Hilbert [9, pp. 370-371], using our u_+ and u_- in place of their v and $-v$. Other recent versions of this method to prove existence for various nonlinear 2nd order elliptic equations can be found in [3] and [8]. Our proof will follow [9, pp. 370-371] suitably generalized to the L_p situation.

Proof. Set

$$k_1(x) = \max(1, -h(x)) ,$$

so that $k_1 \geq 1 > 0$, and $k_1 \geq -h$. Let

$$k(x) = k_1(x)e^{u_+(x)}$$

and observe that $k(x) \geq \text{const} > 0$. Since $u_+ \in H_{2,p}(M)$, by (3.8) we see that u_+ is continuous. Thus $k \in L_p(M)$. We will find the desired solution of (9.1) by iterations. Let

$$L\varphi \equiv \Delta\varphi - k\varphi \text{ and } f(x, u) \equiv c - he^u .$$

Then, using (3.13) we define inductively $u_{j+1} \in H_{2,p}$ as the unique solution of

$$Lu_{j+1} = f(x, u_j) - ku_j ,$$

where $u_0 = u_+$. Note that $u_j \in H_{2,p}$ implies that u_j is continuous, so $f(x, u_j) \in L_p$ and hence $f(x, u_j) - ku_j \in L_p$. Consequently $u_{j+1} \in H_{2,p}$. We claim that

$$(9.4) \quad u_- \leq u_{j+1} \leq u_j \leq \dots \leq u_+ .$$

For example, to prove $u_{j+1} \leq u_j$, one checks inductively that $L(u_{j+1} - u_j) \geq 0$ and then applies the L_p maximum principle (3.15). The other inequalities in (9.4) are proved similarly.

Since u_+ , u_- , and u_j are continuous, inequality (9.4) shows that the $|u_j|$ are uniformly bounded. Consequently, in the L_p norm

$$\|Lu_{j+1}\|_p = \|c - he^{u_j} - ku_j\|_p \leq \text{const} ,$$

so by (3.12), the u_j 's and their first derivatives are uniformly bounded. Therefore, the Arzela-Ascoli Lemma implies that a subsequence of the u_j 's converges uniformly to some continuous function u . In view of the monotonicity (9.4), we conclude that the entire sequence u_j itself converges uniformly to u . Inequality (3.11) then shows that

$$\begin{aligned} \|u_{i+1} - u_{j+1}\|_{2,p} &\leq \text{const} \|L(u_{i+1} - u_{j+1})\|_p \\ &\leq \text{const} (\|h\|_p \|e^{u_i} - e^{u_j}\|_\infty + \|k\|_p \|u_i - u_j\|_\infty) . \end{aligned}$$

Therefore the u_j 's converge strongly in $H_{2,p}$, so $u \in H_{2,p}$. Since $L : H_{2,p} \rightarrow L_p$ is continuous, it follows that u is a solution of (9.1) and satisfies $u_- \leq u \leq u_+$.

Now $u \in H_{2,p}$ so $u \in C^1$. By the Schauder theory (see §3) one proves—inductively—that $u \in C^\infty$ on any open set where $h \in C^\infty$. More precisely, if $h \in C^{j+\alpha}$ then $u \in C^{j+2+\alpha}$. Q.E.D.

Lemma 9.3 shows that the burden of proving existence is shifted to finding a priori estimates of the form (9.2). This is, as usual, the most difficult step. Observe first that one can not expect this to be possible for arbitrary h . In fact, integrating (9.1) over M shows that

$$c \text{ Vol}(M) = \int_M he^u dV .$$

Because $c < 0$, a necessary condition for existence is that h be negative somewhere. This is not sufficient, as we shall see in §10. In the next lemma, we show that given any upper solution u_+ , one can always find a lower solution $u_- \leq u_+$. An extensive discussion of the conditions on h required to obtain an upper solution is presented in §10. We might, however, note that if $\text{const} \leq h \leq \text{const} < 0$, then one can simply use appropriate constants for u_+ and u_- , to prove existence quite easily in this case.

LEMMA 9.5. (Existence of Lower Solutions.) *Given any $h \in L_p(M)$ and a function $u_+ \in H_{2,p}(M)$, there is a lower solution $u_- \in H_{2,p}(M)$ of (9.1) with $c < 0$, such that $u_- \leq u_+$. Thus, a solution exists if (and only if) there is an upper solution.*

Proof. If h is bounded from below, then one can clearly use any sufficiently large negative constant for u_- . For general $h \in L_p$, let $k_1(x) = \max(1, -h(x))$ and let $\alpha > 0$ be a constant chosen so that $\alpha \bar{k}_1 = -c$. Then $(\alpha k_1 + c) = 0$ and $(\alpha k_1 + c) \in L_p$. Thus there is a solution w of $\Delta w = \alpha k_1 + c$. By the L_p regularity theory (see §3), $w \in H_{2,p}$ and hence w is continuous. We claim that by choosing the constant λ sufficiently large, the function $u_- = w - \lambda$ meets our requirements. One can clearly satisfy $u_- \leq u_+$ for any $u_+ \in H_{2,p}$, because w and u_+ are continuous. In addition u_- is a lower solution since

$$\begin{aligned} \Delta u_- - c + h e^{u_-} &= \alpha k_1 + h e^{w-\lambda} \\ &\geq k_1(\alpha - e^{w-\lambda}) > 0 \end{aligned}$$

for λ sufficiently large.

Q.E.D.

10. Analysis of $\Delta u = c - h e^u$ with $c < 0$

In this section we will prove two theorems that collect information concerning conditions under which our equation (9.1) with $c < 0$ has a solution. We do not make any assumptions on $\dim M$.

THEOREM 10.1. *Consider (9.1) with $c < 0$ and $h \in C^\infty(M)$.*

(a) *If a solution exists, then $\bar{h} < 0$. Even more strongly, the unique solution φ of*

$$(10.2) \quad \Delta \varphi + c \varphi = h$$

must be positive.

(b) *If $\bar{h} < 0$, then there is a constant $-\infty \leq c_- < 0$ depending on h (which we write $c_-(h)$) such that one can solve (9.1) for all $c_-(h) < c < 0$, but cannot solve (9.1) if $c < c_-(h)$.**

* In [37, §7A] this is sharpened to nonsolvability if $c \leq c_-(h)$.

Information on the critical constant $c_-(h)$ is supplied by the next theorem in this section.

Proof. (a): Using the substitution $v = e^{-u}$, one sees that (9.1) has a solution u if and only if there is a solution $v > 0$ of

$$(10.3) \quad \Delta v + cv - h - \frac{|\nabla v|^2}{v} = 0 .$$

Assume that v is a positive solution of (10.3), let φ be the unique solution of (10.2) and let $w = \varphi - v$. Then

$$\Delta w + cw = - \frac{|\nabla v|^2}{v} \leq 0 .$$

Since $\Delta w \geq 0$ at a minimum of w , it follows that the minimum of w must be non-negative. Therefore $\varphi \geq v > 0$. Consequently, a necessary condition for there to exist a positive solution of (10.3) (and hence a necessary condition for there to exist a solution of (9.1)) is that the unique solution of (10.2) must be positive. This necessary condition immediately implies $\bar{h} < 0$ as one sees by integrating (10.2) (one also could have proved that $\bar{h} < 0$ by imitating the proof of (5.2)). In the proof of Part (c) of Theorem 10.5 we will find that there may fail to be a positive solution of (10.2) even if $\bar{h} < 0$. Thus the positivity of the solution of (10.2) is a stronger necessary condition than $\bar{h} < 0$. This completes the proof of Part (a).

(b): In view of the two lemmas of §9, a necessary and sufficient condition for there to exist a solution $u \in C^\infty(M)$ of (9.1) is the existence of an upper solution $u_+ \in C^\infty(M)$,

$$(10.4) \quad \Delta u_+ \leq c - he^{u_+} .$$

Clearly, if u_+ is an upper solution for a given $c < 0$, then u_+ is also an upper solution for all $\tilde{c} < 0$ such that $c \leq \tilde{c}$. Therefore, there is a constant $-\infty \leq c_-(h) \leq 0$ such that (9.1) is solvable for negative c 's with $c > c_-(h)$ but has no solutions for $c < c_-(h)$.

We claim that under the assumption $\bar{h} < 0$, we have $c_-(h) < 0$. Indeed, let $v \in C^\infty(M)$ be a solution of $\Delta v = \bar{h} - h$. Since $|e^t - 1| \leq |t|e^{|t|}$, and since $\bar{h} < 0$, we can pick $a > 0$ so small that

$$|e^{av} - 1| \leq \frac{-h}{2\|\bar{h}\|_\infty} .$$

Let $e^b = a$. If $c = a\bar{h}/2$ and $u_+ = av + b$, we have

$$\begin{aligned} \Delta u_+ - c + he^{u_+} &= ah(e^{av} - 1) + \frac{a\bar{h}}{2} \\ &\leq a \|h\|_\infty |e^{av} - 1| + \frac{a\bar{h}}{2} \\ &\leq \frac{a\bar{h}}{2} - \frac{a\bar{h}}{2} = 0. \end{aligned}$$

Thus, with $c = a\bar{h}/2 < 0$, we have an upper solution u_+ . Consequently, $\bar{h} < 0$ implies that $c_-(h) \leq a\bar{h}/2 < 0$. Q.E.D.

Next we discuss the critical constant $c_-(h)$ of Theorem 10.1.

THEOREM 10.5. *Except in (d), we assume that $h \in C^\infty(M)$.*

(a) $c_-(h) = -\infty$ if and only if $h(x) \leq 0$ for all $x \in M$ but $h \not\equiv 0$.

(b) If $\tilde{h} \leq h$, then $c_-(\tilde{h}) \leq c_-(h)$. Also, $c_-(h) = c_-(\lambda h)$ for any constant $\lambda > 0$.

(c) Given $c < 0$, there is an h with $\bar{h} < 0$ such that $c < c_-(h)$. Thus, the necessary condition $\bar{h} < 0$ is not sufficient for solvability of (9.1) and the critical constant $c_-(h) < 0$ can be made arbitrarily close to 0.

(d) Let $h \in L_p(M)$ for some $p > \dim M$. If there exists a constant $\alpha < 0$ and $f \in L_p(M)$ such that $h \leq f$ and $\|f - \alpha\|_p < -\alpha/\gamma(1 - 2c)$, where γ is the constant in (3.12) with $L = \Delta + c$, then there is a solution $u \in H_{2,p}(M)$ of (9.1) with u smooth on any open set in which h is smooth.

The vital part of this theorem for our geometric applications is the somewhat complicated Part (d), which asserts that for given c one can solve (9.1) for a fairly rich class of functions h , in particular for h 's that are occasionally positive, despite the difficulties asserted in (a) and (c). One can show that the condition on h in (d) implies the necessary condition $\bar{h} < 0$ (use (3.12) with $u \equiv 1$ to show first that $1 \leq \gamma c \text{Vol}(M)^{1/p}$).

Proof. (a): First we show that if $h(x) \leq 0$ for all $x \in M$, but $h \not\equiv 0$, then (9.1) is solvable for all $c < 0$. As we have observed before, in view of the lemmas in §9, the solvability of (9.1) is equivalent to the existence of an upper solution, u_+ , of (9.1). Let $\Delta v = \bar{h} - h$, and note that $\bar{h} < 0$. Pick constants a and b so large that $a\bar{h} < c$ and $(e^{av+b} - a) > 0$. Then let $u_+ = av + b$. Since $h \leq 0$, we find that

$$\Delta u_+ - c + he^{u_+} = a\bar{h} - ah - c + he^{av+b} < 0.$$

Therefore u_+ is an upper solution. Consequently $c_-(h) = -\infty$ if $h \leq 0$ but $h \not\equiv 0$.

Secondly, suppose that $h(x_0) > 0$ for some $x_0 \in M$. By the Asymptotic

Theorem 4.4, the unique solution of (10.2) is negative at x_0 for $c < 0$ sufficiently large negative. But by Theorem 10.1 (a) the positivity of this solution of (10.2) is a necessary condition for the solvability of (9.1). Therefore, if h is positive somewhere, then (9.1) has no solution for c large enough negative.

(b): If u_+ is an upper solution for h , then it certainly is an upper solution for any $\tilde{h} \leq h$. Therefore $c_-(\tilde{h}) \leq c_-(h)$. To see that $c_-(h) = c_-(\lambda h)$ for any constant $\lambda > 0$, we merely note that if u is a solution of $\Delta u = c - he^u$, then $v = u - \log \lambda$ is a solution of $\Delta v = c - \lambda he^v$. So for prescribed $c < 0$ one can solve given the function λh if (and only if) one can solve given the function h .

(c): With $c < 0$ given, we show there is an $h \in C^\infty(M)$ with $\bar{h} < 0$ for which (9.1) is not solvable; thus $c < c_-(h) < 0$. Let $\psi \in C^\infty(M)$ satisfy $\bar{\psi} = 0$, but $\psi \not\equiv 0$. Choose a constant $\alpha > 0$ so small that $\psi + \alpha$ still changes sign. Let $h = \Delta\psi + c(\psi + \alpha)$. Then $\bar{h} = c\alpha < 0$. By Theorem 10.1(a) we conclude that (9.1) has no solution for this h and this $c < 0$ since the unique solution of (10.2),

$$\Delta\varphi + c\varphi = h,$$

is $\varphi = \psi + \alpha$, which changes sign.

(d): In view of the lemmas in §9, in order to prove existence for (9.1) it is sufficient to prove there is an upper solution $u_+ \in H_{2,p}(M)$ of (9.1), that is,

$$(10.6) \quad \Delta u_+ - c + he^{u_+} \leq 0.$$

Make the change of variable $v = e^{-u_+}$. Then u_+ will satisfy (10.6) if $v \in H_{2,p}(M)$ is a *positive* solution of

$$(10.7) \quad \Delta v + cv - h - \frac{|\nabla v|^2}{v} \geq 0.$$

Let $\delta = -\alpha/(1-2c)$. We claim that we can let v be the unique solution of

$$(10.8) \quad Lv \equiv \Delta v + cv = f + \delta.$$

Note that $v \in H_{2,p}(M)$, by (3.13), and observe that $\psi \equiv 2\delta$ is a solution of $L\psi = 2c\delta$. Thus $L(v - \psi) = f + \delta(1-2c) = f - \alpha$, so inequality (3.12) applied to $v - \psi$ reveals that

$$(10.9) \quad \|v - 2\delta\|_\infty + \|\nabla v\|_\infty \leq \gamma \|L(v - \psi)\|_p = \gamma \|f - \alpha\|_p < \delta.$$

In particular,

$$\|v - 2\delta\|_\infty < \delta \quad \text{so} \quad v(x) > \delta.$$

Also, $\|\nabla v\|_\infty < \delta$ so

$$(10.10) \quad \frac{|\nabla v|}{v} < 1 \quad \text{and} \quad \frac{|\nabla v|^2}{v} < \delta .$$

Therefore $v(x) > \delta > 0$ and (10.7) is satisfied since

$$\Delta v + cv - h - \frac{|\nabla v|^2}{v} \geq f + \delta - h - \delta = f - h \geq 0. \quad \text{Q.E.D.}$$

Remark 10.11. For later use in [17] we need to observe that under hypothesis (d) of Theorem 10.5 there is a solution $v \in H_{2,p}(M)$ of

$$\Delta v + cv - h - \frac{|\nabla v|^2}{v} \geq 0$$

satisfying $v > 0$ and $(|\nabla v|/v) < 1$. This follows immediately from (10.10).

Remark 10.12. By a maximum principle argument [7, pp. 283-284], one can show that if $h \leq 0$ ($\neq 0$), then the solution to (9.1) is unique.

COROLLARY 10.13. *Assume $f \in C^\alpha(M)$ satisfies $\bar{f} < 0$. Given any $h \in C^\alpha(M)$, there is a diffeomorphism φ of M such that one can find a solution $u \in C^{2+\alpha}(M)$ of*

$$(10.14) \quad \Delta u = f - (h \circ \varphi)e^u$$

if and only if h is negative somewhere. Moreover, if $h \leq 0$ ($\neq 0$), then one can let φ be the identity map.

Proof. Following the reduction (2.1) to (2.3), equation (10.14) is equivalent to the equation $\Delta w = \bar{f} - (h \circ \varphi)e^v e^w$ where $\Delta v = f - \bar{f}$. If $h \leq 0$ ($\neq 0$), the corollary follows from Theorem 10.5(a). If h is positive somewhere, the corollary is proved by showing that there is a diffeomorphism φ such that Theorem 10.5(d) applies. The details of this proof are the same as in Theorem 11.6 below and so are omitted here.

11. Curvatures of compact M with $\chi(M) < 0$

Throughout this section, M will denote a fixed compact connected differentiable manifold of dimension two having negative Euler characteristic, $\chi(M) < 0$. Given a function $K \in C^\infty(M)$, we shall determine when K is the Gaussian curvature of some Riemannian metric \tilde{g} of M (Theorem 11.8), as well as when such a metric is either pointwise conformal or conformally equivalent to some prescribed metric g on M (Theorems 11.1 and 11.6).

Our first theorem deals with the pointwise conformal case. Given a smooth metric g on M , let $PC(g)$ denote the set of functions $K \in C^\infty(M)$ that are the Gaussian curvatures of metrics which are pointwise conformal to g , i.e. metrics of the form $\tilde{g} = e^u g$ for some $u \in C^\infty(M)$. Also, let k and Δ be

the Gaussian curvature and Laplacian, respectively, of the prescribed metric g .

THEOREM 11.1.

(a) *If $K \in PC(g)$, then*

$$(11.2) \quad \int_M Ke^{2v} dA < 0 ,$$

where v is some solution of $\Delta v = k - \bar{k}$.

(b) *Even stronger than (11.2), if $K \in PC(g)$, then the unique solution φ of*

$$\Delta\varphi + 2\bar{k}\varphi = 2Ke^{2v}$$

must be positive.

(c) *There exist $K \in C^\infty(M)$ satisfying (11.2) such that $K \notin PC(g)$ so (11.2) is not sufficient.*

(d) *$K \in PC(g)$ if and only if there is a solution of the differential inequality*

$$\Delta u \leq k - Ke^{2u} .$$

Remark. Inequality (11.2) should not be confused with the more obvious Gauss-Bonnet necessary condition

$$(11.3) \quad \int_M Ke^{2u} dA = 2\pi\chi(M) < 0 .$$

Since u is not a priori known, all that (11.3) states is that K must be negative somewhere. On the other hand, the function v is known so (11.2) is a stronger statement that implies K must be negative somewhere. In the special case of $k \equiv \text{const}$, then $v \equiv \text{const}$ and (11.2) becomes $\bar{K} < 0$, which is evidently quite different from the less tangible (11.3).

Proof. As stated in §1, $K \in PC(g)$ if there is a solution u of

$$(11.4) \quad \Delta u = k - Ke^{2u} .$$

Moreover the change of variable $w = 2(u - v)$, as in (2.1)–(2.2), shows that we need only solve

$$(11.5) \quad \Delta w = 2\bar{k} - (2Ke^{2v})e^w ,$$

where $\bar{k} \equiv \text{const} < 0$ (since $\chi(M) < 0$). Thus the results of §§9–10 are applicable. It is immediately apparent that (a) and (b) follow from Theorem 10.1(a), that (c) follows from Theorem 10.5(c), and that (d) is a consequence of the two lemmas in §9. Q.E.D.

Remark. By the remark after the proof of Theorem 10.5, if $K \leq 0$ ($\neq 0$),

then the pointwise conformal metric \tilde{g} is uniquely determined.

If one only asks that K be the curvature of a metric \tilde{g} conformally equivalent to g , then the situation is much less complicated for our present case of $\chi(M) < 0$.

THEOREM 11.6. *A function $K \in C^\infty(M)$ is the curvature of a metric \tilde{g} that is conformally equivalent to a prescribed metric g if and only if K is negative somewhere.*

Proof. The necessity has been observed in (1.2) as a consequence of the Gauss-Bonnet theorem. For the sufficiency, given such K , we must find a diffeomorphism φ of M and a function $u \in C^\infty(M)$ such that the pulled-back metric $\tilde{g} \equiv (\varphi^{-1})^*(e^u g)$ has curvature K , i.e., for some φ and u , $K \circ \varphi$ is the curvature of the metric $g_1 = e^u g$. In terms of differential equations, we must find a φ such that one can solve

$$\Delta u = k - (K \circ \varphi)e^{2u} .$$

By the change of variables of (11.5), and in view of Theorem 10.5(d), it is sufficient to prove there is a constant $\alpha < 0$, a function $f \in C^\infty(M)$, and a diffeomorphism φ of M such that

$$(11.7) \quad 2(K \circ \varphi)e^{2v} \leq f, \quad \text{and} \quad \|f - \alpha\|_p < |\alpha|/\gamma(1 - 4\bar{k}),$$

where γ is the constant in (3.12) and $p > \dim M = 2$, say $p = 3$.

We now obtain α , f , and φ . If $K < 0$, then $\alpha < 0$ can be chosen so that $\max 2Ke^{2v} \leq \alpha$; then let $f \equiv \alpha$ and φ be the identity map and we are done. Otherwise we let $m = \min(2K)$ and note that $m < 0$. Let $b = \min e^{2v}$, and choose any α such that $mb < \alpha < 0$. We take φ to be a diffeomorphism that makes $2K \circ \varphi$ nearly equal to m on most of M . More precisely, one can find open sets U and V with $\bar{V} \subset U \subset M$ and a diffeomorphism φ of M such that

$$2(K \circ \varphi)e^{2v} < \alpha \quad \text{on } U,$$

and

$$\text{measure}(M - V) < \left(\frac{|\alpha|}{\gamma(1 - 4\bar{k})(\|2K\|_\infty \|e^{2v}\|_\infty - \alpha)} \right)^3 .$$

Let f be any C^∞ function satisfying

$$\begin{aligned} f(x) &\equiv \alpha, & x \in V, \\ \alpha &\leq f(x) \leq \max [2(K \circ \varphi)e^{2v}], & x \in U - V, \end{aligned}$$

and

$$f(x) \equiv \max [2(K \circ \varphi)e^{2v}], \quad x \in M - U .$$

Then one easily verifies that this choice of α , f , and φ satisfies properties (11.7) with $p = 3$. This completes the proof of Theorem 11.6. Q.E.D.

An immediate consequence of Theorem 11.6 is

THEOREM 11.8. *If $\chi(M) < 0$, then a function $K \in C^\infty(M)$ is the Gaussian curvature of a metric on M if and only if K is negative somewhere.*

Remark. There are many examples where the metric g found in Theorem 11.8 is not unique. For example, if $K \equiv \text{constant}$ in some disk $D \subset M$ and if φ is a diffeomorphism of M that leaves $M - D$ fixed, then K is also the curvature of φ^*g . Non-uniqueness of g occurs in yet another manner if M is orientable, for say the metrics g_1 and g_2 give distinct complex structures on M . By Theorem 11.6, there are diffeomorphisms φ and ψ of M such that K is the curvature of *both* of the pulled-back metrics $\tilde{g}_1 = (\varphi^{-1})^*(g_1)$ and $\tilde{g}_2 = (\psi^{-1})^*(g_2)$. Since \tilde{g}_j gives the same complex structure as g_j , we find that $\tilde{g}_1 \neq \tilde{g}_2$, although both \tilde{g}_1 and \tilde{g}_2 have the same curvature K .

VI. Remarks

A. By using Theorem 8.8, we can prove that *in the case of S^2 with the standard metric, the equation $\Delta u = f - e^{2u}$ has no solution for some smooth function f with $\int_M f dA = 4\pi$* . To see this, let K be a positive function for which one can *not* solve $\Delta w = 1 - Ke^{2w}$ (use Theorem 8.8 to find K) and define v by $e^{2v} = K$. Then with $u \equiv v + w$, and $f \equiv 1 + \Delta v$ one finds $\Delta u = f - e^{2u}$ which has no solution. By adding a sufficiently large positive constant to K , one can even insure that $f > 0$ everywhere.

This sheds some light on one possible approach to a proof of the Uniformization Theorem for Riemann surfaces, which, among other things, asserts that any two Riemannian structures on S^2 are conformally equivalent. In particular, each metric g is conformally equivalent to the standard metric. This means that g is pointwise conformally equivalent to some metric of constant curvature 1, i.e., there exists a solution of

$$(R.1) \quad \Delta_g u = k - e^{2u}$$

where k is the curvature of the metric g .

In view of the above non-existence result for $\Delta u = f - e^{2u}$ (with the standard metric), any attempt to prove the Uniformization Theorem by solving (R.1) must critically use the fact that k is the curvature of the metric g . This is a key reason why the attempted proof of the Uniformization Theorem in [5, p. 17–18] for the case of S^2 breaks down.

We incidentally remark that using the Uniformization Theorem and our

Theorem 8.8, one can show that given any metric g on S^2 with curvature k , there are functions K (positive somewhere) for which one cannot solve $\Delta u = k - Ke^{2u}$, where the Laplacian is in the g metric.

B. In our analysis of $\Delta u = +c - he^u$, $c < 0$, we found it helpful to look at the ordinary differential equation

$$(R.2) \quad u' = +c - he^u \quad \text{on } S^1,$$

where S^1 is the periodic interval $0 \leq x \leq 1$. The substitution $v = e^{-u}$ changes (R.2) to the much simpler

$$(R.3) \quad v' + cv = +h, \quad v > 0,$$

so we seek a positive periodic solution. The unique solution of (R.3) is

$$(R.4) \quad v(x) = \frac{1}{e^c - 1} \int_0^1 e^{+ct} h(x+t) dt,$$

from which one can easily deduce most of the assertions corresponding to those in Theorems 10.1 and 10.5. The Asymptotic Theorem 4.4, $\lim_{c \rightarrow -\infty} cv(x) = h(x)$, is also easy. In addition, one can see that if $h < 0$ somewhere, then there is a diffeomorphism φ of S^1 such that the solution v of (R.3), with h replaced by $h \circ \varphi$, is positive. We used this simple example (R.3) as a guide to many of the results in this paper.

C. Our equation (2.3) is of the form

$$(R.5) \quad Lu = f(x, u), \quad x \in M,$$

where L is a linear elliptic operator. If L is invertible and $f(x, s)$ is bounded for all $x \in M$, $s \in \mathbf{R}$, then it is easy to prove existence of a solution to (R.5). If L has a non-trivial kernel but one still assumes f is bounded as well as satisfying certain other conditions, then Landesman-Lazer [21] have proved recent interesting results (see also the simplification and generalization by Nirenberg [24]). Unfortunately, these results do not apply to our equation $\Delta u = c - he^u$ since not only does $L = \Delta$ have a non-trivial kernel, but also $f(x, s) = c - he^s$ is unbounded as $s \rightarrow \infty$. We have been able to apply the techniques in this paper to second order elliptic problems [18]. Our results extend, unify, and clarify a number of seemingly diverse phenomena.

Added in proof: We have recently completely resolved Questions 1 and 3, and have a unified proof showing that the answers to both questions are “yes” for any compact two dimensional manifold [37]. However, this new proof does not yield our results on Question 2 concerning the existence or non-existence of solutions to (1.3).

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Added in proof:

- [37] JERRY L. KAZDAN and F. W. WARNER, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures, (to appear in *Ann. of Math.*).

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