FINAL EXAMINATION

Math. 410 — Spring, 1970

Dr. Kazdan — Due: Fri. May 1 at 4:30 P.M.

DIRECTIONS: Take home, any references allowed. Please do not consult other people. Answer any 7 (out of 12) questions. 15 points each. NO extra credit for more. If more than 7 problems are done, only the first 7 will be graded. Be neat, using only one problem per page and do not use the back of the paper. Computational errors will be severely penalized.

If you wish to find out your grade, hand in a stamped, addressed post card. The exams will not be returned.

1. Find all complex solutions of $e^z = 1 + 2i$.

2. a). Show in a diagram the image in the w plane of the square $\{0 \le x \le 1,\ 0 \le y \le 1\}$ under the map $w = e^{\pi z}$.

   b). Find an analytic function f from the $z = x + iy$ plane to the $w = u + iv$ plane that maps the given region A in the z plane onto the region B in the w plane.

	A	B
i)	$\{x \ge 1\}$	$\{v \ge 0\}$
ii)	first quadrant	$\{u \ge 2\} \cap \{v \ge 3\}$
iii)	$\{x \ge 0\}$	first quadrant

3. Same directions as in #2b (above).

i).	$\{1 \le x \le 2\}$	$\{1 \le u \le 3\}$
ii).	$\{\lvert z\rvert \le 1\}$	$\{\lvert w - 1\rvert \ge 2\}$

4. Evaluate
$$\frac{1}{2\pi i}\int_C \frac{\sin \pi z}{z^2 - 4z + 3}\,dz ,$$
where C is the contour (taken counterclockwise)
   a) $\{\lvert z\rvert = 1\}$
   b) $\{\lvert z - 2\rvert = 3\}$

5. Prove there is no function f analytic in $\{\lvert z\rvert \le 1\}$ having all of the following properties:
   a) $\lvert f(z)\rvert \le 2$ on $\{\lvert z\rvert = 1\}$,
   b) $f(1/3) = 0$,
   c) $f(1/4) = 1$.

6. Let
$$f(z) = \frac{z}{e^z - 1} = \sum_0^\infty B_n z^n$$
be the Taylor series expansion of f in a neighborhood of $z = 0$.
   a) Find the radius of convergence of the series.
   b). Prove that $B_3 = B_5 = B_7 = \dots = B_{2k+1} = 0$, $k = 1, 2, \dots$.

7. Let f be an entire analytic function such that the image of f does not contain the disc $\{\lvert w - a\rvert \le R\}$ for some $R > 0$ and some complex number a. Prove that $f(z) \equiv$ constant. (Suggestion: consider $(f(z) - a)^{-1}$).

8. Find a conformal map of the half plane $\{y \le 1\}$ into the unit disc $\{\lvert w\rvert \le 1\}$ with $0 \mapsto 0$ and $1 \mapsto 1$.

9. Let f be analytic in $\{\lvert z\rvert \le 1\}$ and have the properties
   a) $f(0) = 0$
   b) $\mathrm{Re}\{f(z)\} \le 1$ for $\{\lvert z\rvert = 1\}$.

   Prove that $\lvert f(z)\rvert \le 2$ in $\{\lvert z\rvert \le \frac{1}{2}\}$. (Suggestion: consider $g(z) = h(f(z))$, where h is the map $h(w) = w/(w-2)$ of the half plane $\mathrm{Re}\{w\} \le 1$ into the unit disc with $0 \to 0$).

10. a) Consider an infinite series of the form $\sum_0^\infty a_n z^{-n}$. If this series converges at a point $z = c$, prove that it converges absolutely for all z with $\lvert z\rvert > \lvert c\rvert$.

    b) Consider an infinite series of the form $\sum_{-\infty}^{\infty} a_n z^n$. Prove that this series converges in some annulus $\{r < \lvert z\rvert < R\}$ where r, R may be zero or infinite, and where the inequalities $r < \lvert z\rvert$ and $\lvert z\rvert < R$ may be equality for some series.

11. a) If f is analytic in $\{\lvert z\rvert \le R\}$ and $\lvert f(z)\rvert \le M$ for $\lvert z\rvert = R$, show that
$$\lvert f(z) - f(0)\rvert \le 2M\frac{\lvert z\rvert}{R} .$$
    b) Use this result to give a proof of Liouville's Theorem.

12. Let f be analytic in $\{|z| \leq 1\}$ and have the properties
    a) $f(0) = 0$,
    b) $f(z) \neq 0$ for $z \neq 0$,
    c) $|f(z)| = 1$ for <u>all</u> z with $|z| = 1$.

Prove that f must be of the form $f(z) = c\, z^n$, for some positive integer n and some constant c, $|c| = 1$.

BONUS PROBLEM If this problem is done, you need only do one (1) other problem, making a total of 2 instead of 7 problems.

Let f be an entire analytic function with the properties
  a) $f(x + 2\pi) = f(x)$, where x is any real number
  b) $|f(z)| \leq e^{a|z|}$ for all z.

Prove that f has the form

$$f(z) = \sum_{k=-n}^{k=n} a_k e^{ikz}, \quad \text{where } n \leq a.$$

Complex Analysis Problems

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1. Assume $f=u+iv$ is analytic in $\{|z|\le R\}$.

   a) Show that $\quad f(z)+\overline{f(0)}=\dfrac{1}{i\pi}\oint_{|\zeta|=R}\dfrac{u(\zeta)}{\zeta-z}\,d\zeta.$

   b) Find a formula determining v inside the disk in terms of u on the boundary of the disk.

   c) Show that $\quad |f(z)|\le\dfrac{2|z|}{R-|z|}A(R)+|f(0)|, \quad$ where $A(R)=\sup_{|z|=R}|u(z)|$.

   d) More generally, $\quad |f^{(k)}(z)|\le\dfrac{2k!R}{(R-|z|)^{k+1}}A(R)+|f(0)|\;$ for $k=1,2,\ldots$.

2. Let $f(z)$ be analytic in the disk $\{|z|<r\}$ and assume $f(0)=0$. Prove that the series $\sum_0^\infty f(z)^n$ converges for $|z|<r$.

3. let $D\subset\mathbb{C}$ be open and connected and $f_n:D\to\mathbb{C}$ be bounded analytic functions. If $f_n(z)\to z$ uniformly in D, prove that for some N we have $n>N$ implies f_n is univalent.

4. Let $a_1,\ldots,a_n$ be arbitrary distinct points in a domain $D\subset\mathbb{C}$. Show there is a function f that is analytic in D such that f is univalent and $f(a_j)=a_j$.

   [SUGGESTION: Consider $f_k(z)=z+\dfrac{(z-a_1)+\cdots+(z-a_k)}{k}$ for large k].

5. Use residues to evaluate (a). $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$, $\qquad$ (b). $\displaystyle\int_0^\infty\frac{\sin^2 x}{x^2}\,dx$.

6. If $f(z)$ is analytic in a neighborhood of $z=c$ show that $h(z):=\overline{f(\bar z)}$ is analytic in a neighborhood of $z=\bar c$.

7. Let a function $f(z)$ have all of the properties

   (a). analytic in $\{|z|\le 1\}$,

   (b). $|f(z)|=1$ for $|z|=1$,

   (c). $f(a)=0$ for some $|a|<1$,

   (d). $f(z)\ne 0$ for all $|z|\le 1$, with $z\ne a$.

   Prove that $f(z)=\alpha\left(\dfrac{z-a}{1-\bar a z}\right)^n$, where $|\alpha|=1$ and n is some positive integer.

8. Let $f(z)$ be analytic in $\{|z|\le 1\}$. If $f(e^{i\theta})=0$ for $a\le\theta\le b$ (where $a<b$), prove that $f\equiv 0$. [In fact, if $f(z)=0$ on a set of positive measure on $\{|z|=1\}$, then $f\equiv 0$. This is the F. and M. Riesz Theorem.]

9. Let H_2 be the normed linear space of functions analytic in $\{|z|<1\}$, $f(z)=\sum_{n\ge 0}a_nz^n$ with finite norm $\|f\|^2=\sum_{n\ge 0}|a_n|^2$.

   a) Prove that H_2 is a (complete) Hilbert space. Prove that H_2 is a ring with the usual pointwise product: $(fg)(z):=f(z)g(z)$.

   b) If $|\alpha|<1$, let $\Phi(z):=\dfrac{z-\alpha}{1-\bar\alpha z}$. Show that $\Phi(z)\in H_2$ and compute its norm.

   c) Show that the functions $e_0:=\Phi(z)$, $e_1:=z\Phi(z),\ldots,e_k:=z^k\Phi(z)$ are orthonormal in H_2. Is this a *complete* orthonormal set?

   d) Define the linear map $M:H_2\to H_2$ by $(Mf)(z):=\Phi(z)f(z)$. Show that indeed $Mf\in H_2$; in fact, M is an isometry into H_2. Prove that the image of M is a closed ideal of the ring H_2. More precisely, the image of M are those functions $g\in H_2$ with $g(\alpha)=0$.

   e) Prove that in the H_2 inner product, for any $f\in H_2$: $f(\zeta)=\langle f(z),(1-\bar\zeta z)^{-1}\rangle$ for all $|\zeta|<1$.

10. Let $f(z):=z+a_2z^2+a_3z^3+\cdots$ be analytic in $\{|z|\le 1\}$. If the a_k's are all *real* and f is univalent, prove that $|a_n|\le n$ by the following steps.

    a) If $f=u+iv$, show that $v(\bar z)=-v(z)$, so $v(x,-y)=-v(x,y)$. Conclude that $v(re^{i\theta})\ge 0$ for $0<\theta<\pi$, $0\le r<1$.

    b) Show $a_nr^n=\frac{2}{\pi}\int_0^\pi v(re^{i\theta})\sin n\theta\,d\theta$.

    c) Prove and use $|\sin n\theta|\le n|\sin\theta|$ to complete the proof.

    d) If $|a_N|=N$ for some N, find f.

11. Let $f=u+iv$ be analytic in some connected open set $\Omega\subset\mathbb{C}$ and let $P(\xi,\eta)$ be a polynomial (in two variables). If $P(u,v)$ is identically zero on Ω, prove that f must be a constant. [SUGGESTION: Observe the hypothesis can be stated as $Q(f,\bar f)\equiv 0$ for some polynomial Q.]

12. If f is analytic in $\{|z|<r\}$ and continuous in $\{|z|\le r\}$, prove:
   a) PARSEVAL'S IDENTITY: $\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta = \sum_{n=0}^{\infty}|a_n|^2 r^{2n}$.
   b) $\sum_{n=0}^{\infty}|a_n|^2 r^{2n} \le M^2(r)$, where $M(r)=\max_{|z|=r}|f(z)|$.
   c) CAUCHY INEQUALITIES: $|a_n|\le \frac{M(r)}{r^n}$.
   d) MAXIMUM PRINCIPLE: If $|f(0)|=M(r)$ for some $r>0$, then f is a constant.
   e) If $a_1,\dots,a_n$ are any complex constants, then $\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{a_j\bar{a}_k}{1+j+k}\le \pi\sum_{\ell=1}^{n}|a_\ell|^2$.
   [SUGGESTION: Consider $f(z)=a_0+a_1z+\cdots a_nz^n$.]

13. Let a_k be the Fibonacci sequence: $a_0=a_1=1$, $a_{n+2}=a_{n+1}+a_n$. Show that

$$\sum_{n=0}^{\infty}a_nz^n=\frac{1}{1-z-z^2}.$$

14. Let f be an analytic function for $r<|z|<R$ and assume that $|f(z)|\le M$.
   a) Show that the Laurent coefficients a_{-n} about $z=0$ satisfy $|a_{-n}|\le Mr^n$, $n=0,1,2,\dots$.
   b) As a special case, if f is analytic function for $0<|z|<R$ and $|f(z)|\le M$, show that $a_{-n}=0$, $n=1,2,\dots$ and hence that $f(z)$ has a removable singularity at $z=0$.
   c) Show the conclusion of part b) still holds if the assumption "$|f(z)|\le M$" is weakened to $|f(z)|\le M|z|^\alpha$ for some $\alpha>-1$.

15. Let $\alpha_j\in\mathbb{C}$, $j=1,2\dots$ satisfy $|\alpha_j|<|\alpha_{j+1}|\to\infty$ and say $A_j\in\mathbb{C}$ are such that $\sum_1^\infty|A_j/\alpha_j|$ converges. Show that the function $f(z):=\sum_1^\infty\frac{A_j}{z-\alpha_j}$ is analytic everywhere except for simple poles at the α_j.
   [SUGGESTION: Show that the series $\sum_{j\ge p}\frac{A_j}{z-\alpha_j}$ converges uniformly for $|z|<|\alpha_p|/2$].

16. Let $P(z)$ be a polynomial. Prove that the zeroes of $P'(z)$ lie in the smallest convex polygon containing the zeroes of P.

[SUGGESTION: If $a_1,a_2,\dots,a_n$ are the zeroes of P (counted with their multiplicity, then

$$\frac{P'(z)}{P(z)}=\frac{1}{z-a_1}+\dots+\frac{1}{z-a_n}.$$

Note that if $c_j\ge 0$ satisfy $c_1+\dots+c_n=1$, then $z=c_1z_1+\dots+c_nz_n$ if and only if z is in the convex hull of the set determined by $z_1,\dots,z_n$.]

17. a) Let $f(z)=\log\frac{1+z}{1-z}$. Describe $f(\{|z|<1\})$.
   b) Let $f(z)=\log(1-z^2)$. Describe $f(\{y>0\})$.

18. (Generalized Schwarz Lemma). If f is analytic in $\{|z|<1\}$ and $\sup_{|z|<1}|f(z)|\le 1$, prove that for $|z|<1$:

$$\frac{|f(0)|-|z|}{1-|f(0)||z|}\le|f(z)|\le\frac{|f(0)|+|z|}{1+|f(0)||z|}.$$

19. Where does the series $\sum\frac{z^n}{1+z^{2n}}$ converge?

20. Find a smooth function $u(x,y)$ for $y>0$ with *all* of the properties:
   a). $u(x,y)$ is harmonic,
   b). $u(x,y)$ is bounded for $y\ge 0$,
   c). $u(x,y)$ is continuous for $y\ge 0$ — except at the origin,
   d). $u(x,0)=0$ for $x<0$,
   e). $u(x,0)=1$ for $x>0$,
   Also prove there is only one such function.

21. Let $u(x,y)$ be a real harmonic function in a domain $\Omega\in\mathbb{C}$. Find all real harmonic functions $v(x,y)$ so that $\varphi(x,y):=uv$ is also harmonic. In other words, when is the product of two (real) harmonic functions also harmonic?

22. A complex number is a *period point* of a meromorphic function f if $f(z+a)=f(z)$ for all z. Thus the period points of e^z are $z=2n\pi i$, $n=0,\pm1,\pm2,\dots$. Show that the period points of a non-constant meromorphic function cannot have a finite limit point.

23. a) If an entire function $f(z)$ has period both 1 and i, prove that it must be a constant.

b) If f is meromorphic in $\mathbb{C}$ and is doubly periodic with period 1 and $\sqrt{2}$, prove that $f \equiv$ constant.

24. Let w_1, w_2, w_3 be linearly independent complex numbers in the sense that if $n_1 w_1 + n_2 w_2 + n_3 w_3 = 0$ for some *integers* n_1, n_2, n_3, then $n_1 = n_2 = n_3 = 0$. If f is meromorphic in $\mathbb{C}$ and is triply periodic, that is, $f(z + w_j) = f(z)$ for all z, prove that $f =$ constant. Thus, *there are no non-constant triply periodic meromorphic functions.*

25. Let $f(z) = u + iv$ be analytic in $\bar{\Omega}$ where $\Omega \subset \mathbb{C}$ is a simply connected domain. If $f(z) \neq 0$ for $z \in \partial\Omega$ and $u = 0$ at $2n$ points on $\bar{\Omega}$, prove that f has at most n zeroes inside Ω.

26. Let $u(x,y)$ be a real harmonic function in $\bar{\Omega}$, as in the previous problem. If the restriction of u to the boundary, $\partial\Omega$, has precisely n local maxima and n local minima (all non-degenerate), prove that grad u is zero at $n-1$ points in Ω.

   [SUGGESTION: First assume Ω is a disk and observe that the function $g(z)$ defined by

$$g(z) := \tfrac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

   is analytic in Ω (this construction is the simplest way to go from a real harmonic function to an analytic function — see problem 59b) below). Let $h(z) = zg(z)$ and show that $g(z) = ru_r - iu_\theta$, where r and θ are polar coordinates. Apply the principle of the argument to g.]

27. Let $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are polynomials without common zeroes and degree $q \geq 2 +$ degree p. If Γ is a simple closed curve bounding a domain containing all the zeroes of q, show that $\oint_\Gamma f(z)\,dz = 0$.

28. If $f(z)$ is analytic in $\{|z|\} < r$, show that $g(z) := \overline{f(\bar{z})}$ is also analytic there.

29. Let $f(z)$ be an entire analytic function. If there are real constants a, b so that $|f(z)| \leq a + b\log(1+|z|)$ for all $z \in \mathbb{C}$, prove that $f \equiv$ constant.

30. Prove that $f(z) := z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent for $|z| < 1$ if $\sum_{n=2}^{\infty} n|a_n| \leq 1$.

31. Let $f(z) = p(z)/q(z)$, where p and q are polynomials with no common factors and $\deg q \geq \deg p$. Thinking of f as a map from $C \to \mathbb{C}$, a point $\zeta \in \mathbb{C}$ such that $f(\zeta) = \zeta$ is called a *fixed point* of f. Let A be the set of all fixed points of f and assume that at any such fixed point, $f'(\zeta) \neq 1$. Prove

$$\sum_{\zeta \in A} \frac{1}{1 - f'(\zeta)} = 1.$$

   [SUGGESTION: Consider the residues of $(f(z) - z)^{-1}$].

32. If $f(z) = \sum a_n z^n$ is holomorphic in unit disk $\{|z| < 1\}$ and if $|f(z)| \leq \dfrac{1}{1-|z|}$, show that the Taylor coefficients satisfy

$$|a_n| \leq (n+1)(1+\frac{1}{n})^n < e(n+1).$$

33. Let $f(z) = \sum a_n z^n$, where the series is assumed to converge for $|z| < r$ and denote by ω a kth root of unity, so $\omega = e^{2\pi i/k}$. Show that

$$\frac{1}{k}\sum_{\nu=0}^{k} f(\omega^\nu z) = \sum_0^\infty a_{nk} z^{nk}.$$

34. Let $f(z) = \sum_0^{n+1}(a_k \cos kz + b_k \sin kz)$, where the a_k and b_k are real numbers with $a_{n+1} + ib_{n+1} \neq 0$.

   a) Show that $f(z) = g(e^{iz})$, where $g(z) = \sum_{-(n+1)}^{n+1} A_k z^k$ with $A_{-k} = \bar{A}_k$ and $A_{n+1} \neq 0$.

   b) Show that $f(x) = 0$ for $-\pi \leq x < \pi$ exactly $2(n+1)$ times, counting multiplicity of course.

35. Assume f is analytic in the closed unit disk $\{|z| \leq 1\}$. Prove that

$$\int_{-1}^{1} |f(x)|^2 \leq \tfrac{1}{2}\int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta.$$

   [SUGGESTION: Write $f = g + ih$ where $g(z) = \frac{1}{2}[f(z) + \overline{f(\bar{z})}$, $h =$? and note that g and h are real for z real. First prove the inequality for g and h separately by applying the Cauchy theorem separately to the upper-half circle and lower-half circle.]

36. Map the sector $\{z = e^{i\theta},\ 0 < r < 1,\ 0 < \theta < \pi/6\}$ conformally onto the upper half plane.

37. Let $P(\tau) = a_0 + a_1\tau + \cdots + a_m\tau^m$ and $P_j = a_j + a_{j+1}\tau + \cdots + a_m\tau^{m-j}$.

a) Show that $\oint_\gamma \frac{P_{j+1}(\tau)\tau^k}{P(\tau)}\,d\tau = \delta_{jk}$, where the contour γ in the complex plane encloses all the roots of the polynomial P.

b) Let $P(D)$ be the constant coefficient ordinary differential operator

$$P(D) = a_0 + a_1 D + \cdots + a_m D^m, \qquad \text{where } D = d/dt.$$

Show that the solution of $P(D)u = 0$ with initial conditions $D^j u|_{t=0} = c_j$, $j = 0, \ldots, m-1$, is

$$u(t) = c_0 v_0(t) + c_1 v_1(t) + \cdots + c_{m-1} v_{m-1}(t),$$

where

$$v_k(t) = \frac{1}{2\pi i}\int_0^t \mathcal{U}(t-s) f(s)\,ds, \quad \text{with} \quad \mathcal{U}(t) = \frac{1}{2\pi i}\oint_\gamma \frac{e^{\tau t}}{P(\tau)}\,d\tau,$$

and γ is as in part a).

38. Let f be holomorphic in nthe unit disk $\{|z| < 1\}$. Show that

$$f(0) = \frac{1}{\pi r^2}\iint_{|z|<r} f(z)\,dx\,dy \qquad \text{for } 0 < r < 1.$$

39. Find *all* entire functions with simple zeroes at $z = n!$, $n = 1, 2, \ldots$ and no other zeroes.

40. Let $\varphi(x,y)$ be a *real* harmonic function for $x^2 + y^2 < 1$. Assume that $\varphi(x,0) = x$ and $\varphi_y(x,0) = 0$. Compute $\varphi(\frac{i}{2})$.

41. Prove there is no meromorphic function $f = u + iv$ with $v > 0$ and $f'(0) = 1$.

42. Let f be a univalent (that is, one-to-one) conformal map of the first quadrant $\{x > 0.\ y > 0\}$ onto the unit disk $\{|z| < 1\}$. Prove there is precisely one meromorphic function $g : \mathbb{C} \to \mathbb{C}$ that coincides with f on the given quadrant. Show that g has exactly two poles, both of first order, and their mid-point is 0.

43. Locate and classify the singularities of the following functions (please don't forget the point at infinity):

$$(a).\ \frac{z}{1+z^2}, \qquad (b).\ \frac{1+3z^2}{1+z}, \qquad (c).\ \sqrt{1+z}, \qquad (d).\ \sin(1/z).$$

44. Let $f : [-\pi, \pi] \to \mathbb{C}$ be continuous and periodic, $f(\theta + 2\pi) = f(\theta)$ for all $\theta \in [-\pi, \pi]$. Say the Fourier series of f is

$$f(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}.$$

Prove there is a (unique?) function $F(z)$ that is analytic in $\{|z| < 1\}$ and continuous in $\{|z| \le 1\}$ such that $F(e^{i\theta}) = f(\theta)$ if and only if all the negative Fourier coefficients are zero: $a_n = 0$ for $n = -1, -2, \ldots$.

45. Let f be analytic in $\{|z| < 1\}$, and continuous in $\{|z| \le 1\}$ and assume that for some integer n, $\operatorname{Re}\{z^n f(z)\} = 0$ on $|z| = 1$. Prove the following.

a) If $n > 0$ then $f \equiv 0$.

b) If $n = 0$ then $f \equiv$ constant.

c) If $n < 0$ then f is a polynomial (of what degree?).

d) What if f is only meromorphic in $\{|z| < 1\}$, and continuous in $\{|z| \le 1\}$?

46. If f is entire analytic and $|f(z)| \ge 1$ everywhere, what can you conclude? Justify your assertions.

47. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in a neighborhood of $z_0 \in \mathbb{C}$ and assume that $f'(z_0) \ne 0$. Regarding f as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, let J be the Jacobian 2×2 matrix.

a) Prove that $J = PR$, where P is a positive scalar matrix (=positive multiple of the identity matrix) and R is an orthogonal matrix with $\det R = 1$.

b) Use this to prove that an analytic function f is a conformal map at all points z where $f'(z) \ne 0$.

48. Let f be analytic in $\{|z| < 1\}$ with $\sup_{|z|<1} |f(z)| \le M$. If $f(a_1) = f(a_2) = \cdots = f(a_n) = 0$ where $|a_k| < 1$, $k = 1, \ldots, n$, prove that

$$|f(0)| \le M \prod_{k=1}^{n} |a_k|.$$

49. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$, where the series converge for $|z| < R_1$ and $|z| < R_2$, respectively. If $h(z) = \sum_{k=0}^{\infty} a_k b_k z^k$, show that for $|z| \le \rho < R_1 R_2$

$$h(z) = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} f(\zeta) g\left(\frac{z}{\zeta}\right) \frac{dz}{\zeta}.$$