d. Show in a diagram the image, in the w-plane, of the square $\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases}$ in the z-plane $(z = x + i\,y)$ under each of the mappings:

(1) $w = e^{\pi z}$

(2) $w = i\,z + 1$

(3) $w = (i + 1)z$

Mathematics 245 Name____________________

Functions of a Complex Variable

Final Exam (Friday section)

Jan. 22, 1960, 6:00 P.M. to 8:00 P.M.

Please write on these sheets.

I. (8 credits) Express the following numbers in the form a + bi:

$(2 + 5i)(3 + i) =$

$25 \div (4 + 3i) =$

$(1 + i)^4 =$

$e^{i\pi/2} =$

$e^{i\pi/4} =$

$\ln i =$

$\ln(1 + i) =$

$3^i =$

II. (8 credits) Place the appropriate sign $(<, >, \leqq, \geqq, =,$ or $\lesseqgtr)^*$ in the following expressions, where $\alpha, \beta, \ldots$ are complex numbers:

$\lvert\alpha + \beta\rvert$	$\lvert\alpha\rvert + \lvert\beta\rvert$
$\lvert\alpha - \beta\rvert$	$\lvert\alpha\rvert - \lvert\beta\rvert$
$\lvert\alpha\beta\rvert$	$\lvert\alpha\rvert \cdot \lvert\beta\rvert$
$\mathrm{Im}(\alpha + \beta)$	$\mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$
$\alpha\bar{\alpha}$	$\lvert\alpha\rvert^2$

* The last of these symbols means merely that there is no relation between the quantities: either one may be larger or they may be equal.

$|\alpha + \beta|^2$ $\qquad$ $4\mathrm{Re}(\alpha\bar{\beta})$

$\dfrac{1}{|\alpha + \beta|}$ $\qquad$ $|\frac{1}{\alpha}| + |\frac{1}{\beta}|$

$|\alpha + \beta|$ $\qquad$ $|\alpha - \beta|$

III. (14 credits) Complete the following definitions:

1. The series $c_0 + c_1 + \cdots + c_n + \cdots$ is said to be absolutely convergent if

2. The series $\phi_0(z) + \phi_1(z) + \cdots + \phi_n(z) + \cdots$ is said to be uniformly convergent for z in a set S if

3. A function $f(z)$ is said to be analytic in an (open) domain D if

4. A point set in the complex plane is said to be an open set if

5. A point set in the complex plane is said to be simply connected if

6. The function $f(z)$ is said to have an isolated singularity at $z = z_0$ if

7. The function $f(z)$ is said to have a simple pole at $z = z_0$ if

IV. (12 credits) Give the power-series expansions about $z = 0$, and the radii of convergence, for the following functions, using the format of the 1st example:

function	series	radius of conv.
e^z	$= 1 + z + \frac{1}{2} z^2 + \cdots + (\frac{1}{n!})z^n + \cdots$	∞
$\cos z$	$=$	∞
$\dfrac{1}{1 + z}$	$=$	
$\ln(3 + z)$	$=$	
$\sqrt{1 + z^3}$	$=$	

V. (10 credits) For each of the following functions $u(x,y)$, give a function $v(x,y)$ (the so-called harmonic conjugate of $u(x,y)$) such that $u(x,y) + i\, v(x,y)$ is an analytic function of $z = x + i y$ in the unit circle $|z| < 1$; if no such function exists, write "none"

$u(x,y) = x$, $v(x,y) =$

$u(x,y) = x^2 - y^2$, $v(x,y) =$

$u(x,y) = x^2 + y^2$, $v(x,y) =$

$u(x,y) = \ln\sqrt{(x+2)^2 + y^2}$, $v(x,y) =$

$u(x,y) = e^x \cos y$, $v(x,y) =$

VI. (14 credits) Mark all the correct conclusions from the stated premises (there may be more than one) by encircling the corresponding letters (a, b, etc.).

1. If the real-valued functions $u = u(x,y)$ and $v = v(x,y)$ have continuous derivatives satisfying $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ in a domain D, the function $f(x + i y) = u + i v$ is necessarily
   a. continuous in D.
   b. a constant.
   c. analytic in D.
   d. bounded in D.
   e. an entire function.

2. If $f(z)$ is an entire function (i.e. analytic for all

z), and if $|f(z)| < 1$ for $|z| > 1$, then,
   a. $|f(z)| < 1$ for all z
   b. $f(z)$ is a constant
   c. $f(z)$ is real-valued
   d. $f(z) = 0$

3. If $f(z)$ is defined by $\sum_{0(n)}^{\infty} a_n z^n$ whenever this this series converges, and if $|a_n| \leq 1/n!$ for all n, then
   a. $|f(z)| \leq |e^z|$
   b. $f(z)$ is an entire function
   c. $1/f(z)$ is an entire function
   d. $\overline{f(z)}$ is an entire function
   e. $|f(z)| \leq e^{|z|}$

VII. (8 credits) Complete each of the following statements to indicate the values of the complex variable z for which the statement is valid:

(Example: $|e^z| < 1$ for <u>$Re(z) < 0$</u>)

1. If the series $\sum_{0(j)}^{\infty} a_j z^j$ converges for $z = 1 + i$, it converges absolutely for
2. $|e^{iz}| < 2$ for
3. $\sin^2 z + \cos^2 z = 1$ for
4. $Re(\ln z) < 0$ for

VIII. (10 credits) values of the following integrals, the path of in being in each case the circle $|z| = 1$ described cowise:

$\int z\, dz$

$\int \bar{z}\, dz$

$\int \frac{1}{z}\, dz$

$\int \sin(z^2$

$\int \frac{\sin z}{z^2}$

FINAL EXAM
Closed Book

Prof. L. Bers
Math 245 Sec A

January 24, 1962

Answer 6 questions.

1. Find the radius of convergence of the power series $\sum_0^\infty a_n z^n$, given that

(a) $\sum_0^\infty a_n e^{n^2}$ converges

(b) $a_n = \int_{|z|=1} \frac{\cos\left(\frac{1}{z - 10}\right) dz}{z^{n+1}}$

(c) $a_n = \int_{|z|=n} \frac{e^{\sin z}\, dz}{(z - 1000)^{10}}$

2. Given that

$$\frac{\sin z}{z(z^2 - 1)} = \sum_{-\infty}^{+\infty} a_n z^n$$

in some domain, find all possible values of a_{-2}.

3. Compute

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1 + x^2}\, dx$$

4. Given that $f(z)$ is holomorphic for $|z| < \infty$, and that for real x, $-1 < x < 1$, $|f(x)| \leq e^{-1/x^2}$. What can you say about f?

5. Let f(z) be entire. Assume that $f(z) \neq 0$ for $z \neq n$ ($n = 1,2, \ldots$) and that

$$\int_{|z| = n-1/2} \frac{dz}{f(z)} \neq \int_{|z| = n+1/2} \frac{dz}{f(z)}.$$

Is this function transcendental?

6. Find all functions f(z) such that f(z) is holomorphic except perhaps for $z = 0,1,2,3,4,$

$$|f(z)| \leq \left(\frac{100\,|z|}{|z-1|\,|z-2|\,|z-3|\,|z-4|} \right)^{\frac{21}{20}}$$

and

$$\int_{|z|=3/2} f\,dz = 1\ , \quad \int_{|z|=5/2} f\,dz = -3\ ,$$

$$\int_{|z|=7/2} f\,dz = \int_{|z| = 10} f\,dz = 0 \quad .$$

7. Let f(z) be holomorphic for $|z|<2$. Where does the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(z)}{dz^n}$$

converge?

8. Let f(z) be holomorphic for $|z| \leq 1$, $f(0) = 0$, $f(1) = 1$ and $|f(z)| \leq 1$ for $|z| = 1$. Show that $|f'(1)| \geq 1$.

9. State and prove the maximum modulus theorem.

10. State and prove Liouville's theorem.

Professor Nirenberg Math 2450A

FINAL EXAMINATION

Complex Variables

1. Find the radius of convergence of each of the following series

(i) The power series in $(z-1)$ about 1 of the function $e^z + \log z^2$.

(ii) $\sum_0^{\infty} (1 + \cos \frac{n\pi}{4})^n z^n$

(iii) $\sum_0^{\infty} z^n \sin n^2$

2. Find the Laurent series expansion near ∞ in powers of z of the function

$$f(z) = \frac{1}{(z-1)(z-2)} .$$

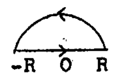
Is it regular at ∞ ?

3. (a) State Rouché's theorem.

(b) Let f(z) be analytic in the strip $|\operatorname{Im} z| < 10$ with $|f(z)| < 1$. Prove that $\cos z + f(z)$ has an infinite number of zeros in the strip.

4. Evaluate the integral

$$\int \frac{\log x}{4+x^2}\,dx .$$

around the half circle  for R large. By taking its real part evaluate

$$\int_0^{\infty} \frac{\log x}{4+x^2}\,dx .$$

5. (a) The function f(z) is analytic in the punctured disc

$$0 < |z| < 1 ,$$

continuous in $0 < |z| \leq 1$, bounded in $|z|<1$, and $|f(z)| \leq 1$ on $|z| = 1$. Prove that $|f(z)| \leq 1$ everywhere.

(b) Assuming furthermore that on $|z| = 1$, $\operatorname{Re} f(z) = y = \operatorname{Im} z$, determine the function f, proving your statement.