Math 609 **Exam 1** Jerry L. Kazdan
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DIRECTIONS This exam has two parts, the first is short answer (*6 points each*) while the second has traditional problems (*12 points each*). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Part A: Short Answer Problems (5 problems, 6 points each)

A–1. Find all complex values of 1^i in the form $a + ib$.

A–2. a) For which values of the constant c is $u(x, y) := 2y + e^{3y} \sin cx$ the real part of an analytic function $f = u + iv$?

b) For these values of c, find the corresponding function $f(z)$.

A–3. If $\sum a_n z^n$ is the power series expansion of $\dfrac{1}{\cos(z + 1)}$ about $z = 0$, what is its radius of convergence?

A–4. Compute $\displaystyle\oint_C \frac{e^z}{z^2 - 2z}\, dz$, where C is the ellipse $x^2/25 + y^2/9 = 1$ (counterclockwise).

A–5. Let $f(z)$ be holomorphic for $0 < |z| < \infty$. If f has no zeroes and $|f(z)| \ge |f(2)|$ in the disk $|z - 2| < 1$, what can you conclude about $f(z)$? Justify your assertions.

Part B: Traditional Problems (6 problems, 12 points each)

B–1. Let $f(z)$ be holomorphic in $\{|z| \le 1\}$ except for a simple pole at $z = i/2$. If f also satisfies $f(\frac{1}{2}) = 0$ as well as $|f(z)| \le 1$ on $|z| = 1$, show that $|f(0)| \le 1$.

B–2. Find a conformal map $f(z) = u + iv$ from the unit disk $\{|z| < 1\}$ to the first quadrant, $\{u > 0,\ v > 0\}$.

B–3. Give a complete clear proof of the fundamental theorem of algebra: *Every nontrivial complex polynomial has at least one root.*

B–4. Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{1 + x^2}\, dx$.

B–5. Let $g(z)$ be holomorphic in the closed unit disk $D = \{|z| \le 1\}$ and assume that $|g(z)| \le 2$ for $|z| = 1$. How many roots does $h(z) := g(z) + 5z^3 - 2$ have in D? As usual, justify your assertions.

B–6. Let $h(z) = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^z}$, where $z = x + iy$, and assume the sequence a_n is bounded, say $|a_n| \le M$. Show that $h(z)$ is holomorphic in the half-plane $\{x > 1\}$.