Math 609 Jerry L. Kazdan
April 28, 2009 10:30–11:50

Complex Analysis Exam II

DIRECTIONS This exam has two parts, Part A has 4 short answer problems (5 points each so 20 points) while Part B has 7 traditional problems, 10 points each so 70 points).
Closed book but you may use one 3×5 card with notes (on both sides).
All contour integrals are assumed to be in the positive sense (counterclockwise).

Short Answer Problems [5 points each (20 points total)]

A1. If $f(z)$ is an entire function with $|f(z)| \geq 1$ everywhere, what can you conclude about f? Justify your assertions.

A2. If $f(z)$ is an entire function and $f(x+2\pi) = f(x)$ for all real x, does $f(z+2\pi) = f(z)$ for all complex z? Proof or counterexample.

A3. The function $\dfrac{z^3-1}{z^2+3z-4}$ has a power series expansion in a neighborhood of the origin. What is its radius of convergence? Justify your assertion.

A4. Assume the entire function $f(z)$ has no zeroes on any of the circles $|z| = n$, $n = 1, 2, 3, \ldots$ and also that

$$\oint_{|z|=n} \frac{1}{f(z)}\, dz \neq \oint_{|z|=n+1} \frac{1}{f(z)}\, dz, \quad n = 1, 2, 3, \ldots.$$

Is this function transcendental? Proof or counterexample.

Traditional Problems [10 points each (70 points total)]

B1. Assume $f(z)$ is meromorphic for all $|z| < \infty$ and satisfies

$$|f(z)| \leq \left(\frac{2|z|}{|z-1|}\right)^{3/2}.$$

What can you conclude about f? Justify your assertions.

B2. Evaluate $A = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx$ where $a > 0$.

B3. a) Let $f(z)$ be holomorphic in $|z| \leq R$ with $|f(z)| \leq M$ on $|z| = R$. Show that

$$|f(z) - f(0)| \leq \frac{2M|z|}{R}$$

b) Use this to give a proof of Liouville's theorem.

B4. If $f(t)$ is piecewise continuous and uniformly bounded for all $t \geq 0$, show that for $Re\{z\} > 0$ the function (Laplace transform)

$$g(z) := \int_0^\infty f(t)e^{-zt}\,dt$$

is holomorphic for $\mathrm{Re}\{z\} > 0$.

B5. Let $f_n(z)$ be a sequence of functions holomorphic in the connected open set Ω and assume they converge uniformly on every compact subset of Ω. Show that the sequence of derivatives $f_n'(z)$ also converges uniformly on every compact subset of Ω.

B6. Find a conformal map from the unbounded region outside the disks $\{|z+1| \leq 1\} \cup \{|z-1| \leq 1\}$ to the upper half plane.

B7. Consider the family of polynomials

$$p(z;t) = z^n + a_{n-1}(t)z^{n-1} + \cdots a_1(t)z + a_0(t),$$

where the coefficients $a_j(t)$ depend continuously on the parameter $t \in [0,1]$. Assume that at $t = 0$ the polynomial $p(z;0)$ has k zeroes (counted with their multiplicity) in the disk $|z-c| < R$ and has no zeroes on the circle $|z-c| = R$.

Show that for all sufficiently small t the polynomial $p(z;t)$ also has k zeroes in $|z-c| < R$.