Open Set in $\mathbb{R}^2$ Whose Area is not Riemann Integrable

Math 508, Fall 2006 **Jerry Kazdan**

This is an example of a bounded open set in $\mathbb{R}^2$ whose area (in the sense of the Riemann Integral) is not defined.

Let $Q = \{(x,y) \in \mathbb{R}^2 | 0 \le x \le 1,\ 0 \le y \le 1\}$ Consider the points (x,y) in the open unit square $\{0 < x < 1,\ 0 < y < 1\}$ where both x and y are rational. This set is countable. Around the k^{th} point put a small open square T_k (contained in Q) whose area is A_k (many of these sets are not disjoint). Pick the areas so small that $\sum A_k < 1/3$. Our desired open set is $T = \Cup_k T_k$. It's closure is the closed unit square Q, so its boundary $= Q - T$. I claim that the area of the boundary is not zero, so the set S is not Jordan measurable (that is, not defined in the sense of the Riemann integral).
In fact, I'll prove directly that T is not defined in the sense of the Riemann integral.
First, a sketch of the General theory of Jordan measure for any bounded set D in $\mathbb{R}^2$.

Since D is bounded, it is contained in some rectangle. Let $\mathcal{P}$ be a partition of this rectangle into a finite number of disjoint rectangles (whose sides can be parallel the axes). These rectangles are of three classes:

- (i) I_k contain only interior points of the set.
- (ii) J_k contain at least one boundary point of the set.
- (iii) E_k contain only exterior points of the set.

The union of the I_k is the "inner (or inscribed) set", while the union of the I_k and J_k is the "outer (or circumscribing) set". For this partition $\mathcal{P}$, let $s(D,\mathcal{P})$ be the sum of the areas of the inner set and $S(D,\mathcal{P})$ the area of the outer set. Clearly $s(D,\mathcal{P}) \le S(D,\mathcal{P})$. If you refine a partition, the rectangles in class (i) or class (iii) cannot change classes. However, the rectangles in class (ii) may go to class (i) or (iii). Thus $s(D,\mathcal{P})$ can *not* decrease while $S(D,\mathcal{P})$ can *not* increase
Let

$$a(D) = \sup s(D,\mathcal{P}) \text{ over all possible partitions (inner area of } D\text{)},$$
$$A(D) = \inf S(D,\mathcal{P}) \text{ over all possible partitions (outer area of } D\text{)}.$$

Of course $a(D) \le A(D)$. If $a(D) = A(D)$, then we say D is *Jordan measurable* with area $a(D)$.

For my bad set T whose closure is the unit square, the outer area is 1: $A(T) = 1$.
To estimate the inner area, pick any partition by a finite number of rectangles. Then

$$s(T, \mathcal{P}) \leq \sum A_k < \frac{1}{3}.$$

Note that since the T_k are not disjoint, in $\sum A_k$ we may have counted some regions more than once, but even so the sum is still less than $1/3$. Consequently

$$a(T) = \sup s(T, P) < \frac{1}{3}.$$

This shows that

$$A(T) - a(T) > 1 - \frac{1}{3} = \frac{2}{3}$$

so T is not Jordan measurable.

Hope this helps.