Math 509: Problem Set 9 (due Thurs. April 5, 2007)

1. Let $\mathcal{H}$ be a Hilbert space and $L : \mathcal{H} \to \mathcal{H}$ a self-adjoint linear map, so $\langle Lv, w\rangle = \langle v, Lw\rangle$ for all v and w in $\mathcal{H}$. Say $\lambda_1, \ldots \lambda_N$ are eigenvalues of L with corresponding orthogonal eigenvectors $e_1, \ldots, e_N$ and let P_N be the orthogonal projectiion into the space spanned by these eigenvectors.

   Show that $LP_N = P_N L$, that is, $L(P_N v) = P_N(Lv)$ for every $v \in \mathcal{H}$

2. Let $f(x)$ an $g(x)$ be (real) Riemann integrable on $[-\pi, \pi]$ with Fourier series $f(x) \sim \sum c_k e^{ikx}$ and $g(x) \sim \sum d_k e^{ikx}$. Compute $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)$ in terms of the c_k and d_k. [HINT: Apply the Parseval equality to the function $f+g$.]

3. Let $Lu := \dfrac{1}{i}\dfrac{du}{dx}$ acting on $C^1(S^1)$, so the functions u are continuously differentiable and periodic with period 2π. With the usual inner product $\langle u, v\rangle = \int_{S^1} u(x)\overline{v(x)}\,dx$,

   a) Show that $\langle L\varphi, \psi\rangle = \langle \varphi, L\psi\rangle$ for all φ, ψ in $C^1(S^1)$.

   b) Find the eigenvalues and corresponding eigenfunctions for L acting on $C^1(S^1)$.

   c) Repeat the two previous parts for the second order differential operator $Mu := L^2 u = -\dfrac{d^2u}{dx^2}$ acting on $C^2(S^1)$.

4. [Math 241 Review] The simplest wave equation with friction has the form $u_{tt} + 2bu_t = u_{xx}$, where $b > 0$ is a constant. Say the string is on $0 \le x \le \pi$ with ends fixed: $u(0,t) = 0$, $u(\pi,t) = 0$. Find the solution $u(x,t)$ is the string is initially plucked at the mid-point, $x = \pi/2$ so

$$u(x,0) = \begin{cases} \frac{2x}{\pi} & \text{for } 0 \le x \le \frac{\pi}{2} \\ \frac{2(\pi - x)}{\pi} & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases} \qquad \text{and} \qquad u_t(x,0) = 0$$

   Remark: For simplicity you may assume $0 < b < 1$.