Math 509: Problem Set 8 (due Thurs. March 29, 2007)

1. a) In $L_2(-3,3)$ with the standard inner product, show that any even function is orthogonal to any odd fucntion (of course assume that the functions are integrable).

   b) If If f has the Fourier series $f \sim \sum A_n e^{inx}$ and if f is a *real* function, show that $A_{-n} = \bar{A}_n$.

2. Show that the polynomials $P_n(x) = \left(\frac{d}{dx}\right)^n (x^2-1)^n$ are orthogonal in $L_2(-1,1)$.

3. The Chebyshev polynomials $T_n(x)$ can be defined recursively by

$$T_0(x) = 1, \quad T_1(x) = x \quad \text{and} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

   a) Compute $T_n(x)$ for $n \le 5$.

   b) Verify that $T_n(\cos\theta) = \cos(n\theta)$.

   c) Show that the $T_n(x)$ are orthogonal in the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\frac{dx}{\sqrt{1-x^2}}$.

4. Find the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ for

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ -1 & \text{for } -\pi \le x < 0. \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & \text{for } 0 < x \le \pi, \\ 0 & \text{for } -\pi \le x \le 0. \end{cases}$$

   Also, for $f(x)$ use Maple to plot $P_5 f(x) = \sum_{k=-5}^{5} c_k e^{ikx}$. [First convert to $\cos kx$ and $\sin kx$].

5. The L_2 convergence of Fourier series $\|f - P_N f\|_{L_2} \to 0$ (this is Parseval's Theorem) when applied to $f(x) = x^2$, $-\pi \le x \le \pi$ gives a certain identity. What is it? [See also Problem Set 7 #6].

6. Let $f(x)$ be continuous and 2π periodic on $-\pi \le x \le \pi$ and let $\sum c_n e^{inx}$ be its Fourier series. If $\sum_{-\infty}^{\infty} |c_n|$ converges, show that the Fourier series converges uniformly – and to $f(x)$.

7. In a Hilbert space say we know that $v_n \to v$ and $w_n \to w$. Show that $\langle v_n, w_n\rangle \to \langle v, w\rangle$. In particular, $\|v_n\| \to \|v\|$.

8. It is often useful to think of the interval $[-\pi, \pi]$ as the circle S^1. On S^1 define the *convolution* $f*g$ by

$$(f*g)(x) = \frac{1}{2\pi}\int_{S^1} f(x-y)(g(y)\,dy.$$

a) Show that $f*g = g*f$.

b) If $f \sim \sum A_n e^{inx}$ and $g(x) \sim \sum B_n e^{inx}$ are the Fourier series of f and g, find the Fourier series of $f*g$ in terms of the A_n and B_n.

9. [STURM-LIOUVILLE PROBLEMS] On the bounded interval $[a, b]$ Consider solutions $u(x)$ of the differential equation

$$-\frac{d^2u}{dx^2} + q(x)u = \lambda u \quad \text{with} \quad u(a) = 0 \text{ and } u(b) = 0.$$

Here $q(x)$ is a fixed real function. Clearly $u(x) \equiv 0$ is a trivial solution. A number λ for which a non-trivial solution $u(x)$ exists is called an *eigenvalue* of the differential operator $Lu = -\frac{d^2u}{dx^2} + q(x)u$, with $u(x)$ the corresponding *eigenfunction*.

a) Find the eigenvalues and corresponding eigenfunctions of the operator $Lu := -u''$ on the interval $[0, \pi]$ (so $q(x) \equiv 0$).

b) Using the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx,$$

show that for any C^2 functions $\phi(x)$ and $\psi(x)$ that are zero at both boundary points $x = a$ and $x = b$ one has

$$\langle L\phi, \psi\rangle = \langle \phi, L\psi\rangle.$$

c) Show that the eigenvalues λ are real.

d) Show that the eigenfunctions u, v associated with *distinct* eigenvalues $\lambda \neq \mu$ are orthogonal.