Math 509, Spring 2007 Jerry L. Kazdan

Math 509: Problem Set 7 (due Thurs. March 22, 2007)

1. For which value(s) of the constant c can the system of equations:

$$\begin{aligned} u(x,y,z) &= x + xyz \\ v(x,y,z) &= y + xy \\ w(x,y,z) &= z + cx + 3z^2 \end{aligned}$$

can be solved for x, y, z as smooth functions of u, v, w near $(1,1,0)$? Justify your assertion(s).

2. Let $y = f(x,u)$ and $z = g(x,u,v)$ be smooth functions with, say, $f(x_0,u_0) = y_0$ and $g(x_0,u_0,v_0) = z_0$.

   a) Under what condition(s) can one eliminate x from these equations to express z as $z = F(y,u,v)$ as a smooth function of y, u, and v near $x = u_0$, $y = y_0$, $v = v_0$?

   b) Assuming this, then compute $\partial z/\partial u$ and $\partial z/\partial y$ in terms of the derivatives of f and g. To make this computation more specific, assume that

$$f_x(x_0,u_0) = 1, \quad f_u(x_0,u_0) = -2, \quad g_x(x_0,u_0,v_0) = -3, \quad ,g_u(x_0,u_0,v_0) = 4 \text{and} \quad g_v(x_0,u_0,v_0) = -2.$$

3. Let $f(x)$, $a \le x \le b$ be a smooth function.

   a) If $f(c) = 0$ for some $a \le c \le b$, show that

$$|f(x)| \le \int_a^b |f'(t)|\,dt \le \sqrt{b-a}\left[\int_a^b |f'(t)|^2\,dt\right]^{1/2}.$$

   and hence, using the uniform norm $\|f\|_{\text{unif}} := \max_{a\le x\le b}|f(x)|$,

$$\|f\|_{\text{unif}} \le \int_a^b |f'(t)|\,dt \le \sqrt{b-a}\left[\int_a^b |f'(t)|^2\,dt\right]^{1/2}.$$

   b) If $\int_a^b f(t)\,dt = 0$ (this replaces the assumption $f(c) = 0$), show that the above inequality still holds.

   c) Use the result of part b) to show that for *any* smooth f

$$\|f\|_{\text{unif}} \le \int_a^b \left[\,|f'(t)| + \frac{1}{b-a}|f(t)|\,\right]dt \le c\left[\int_a^b \left(|f'(t)|^2 + |f(t)|^2\right)\, dt\right]^{1/2},$$

   where c is a constant depending on $b-a$. [Suggestion: Apply the previous part to $g := f - \bar{f}$ where $\bar{f}$ is the average of f over the interval.]

4. Say one has a real normed linear space whose norm comes from an inner product, so $\|x\|^2 = \langle x, x\rangle$. Show that the norm satisfies the *parallelogram identity*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

and interpret this identity geometrically.

5. [Review from Math 241] Let $\mathcal{P}_3$ denote the space of real polynomials of degree at most three with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Find an orthonormal basis for this space.

6. [Review from Math 241] Consider the space of complex-valued continuous functions $C([-\pi,\pi])$ With the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx,$$

and let T_N be the linear space spanned by e^{ikx} for $|k| \le N$. Find the coefficients c_k so that

$$x^2 = \sum_{|k|\le N} c_k e^{ikx} + h(x),$$

where $h(x)$ is orthogonal to T_N.

7. a) Let $f(x) \in C([0,1])$. If $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 0,1,2,\ldots$, show that f must be identically zero.

   b) Let $f(x) \in C([0,1])$. If $\int_0^1 f(x)x^{2k}\,dx = 0$ for all $k = 0,1,2,\ldots$, what can you conclude?

   c) Let $f(x) \in C([-1,1])$. If $\int_{-1}^1 f(x)x^n\,dx = 0$ for all $n = 0,1,2,\ldots$, what can you conclude?

   d) Let $f(x) \in C([-1,1])$. If $\int_{-1}^1 f(x)x^{2k}\,dx = 0$ for all $k = 0,1,2,\ldots$, what can you conclude?

8. a) Compute $\displaystyle\min_{a,b,c}\int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx$.

   b) Compute $\displaystyle\max\int_{-1}^{1} x^3 h(x)\,dx$ where $h \in L^2(-1,1)$ is subject to the restrictions

$$\int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} xh(x)\,dx = \int_{-1}^{1} x^2h(x)\,dx = 0; \quad \int_{-1}^{1} |h(x)|^2\,dx = 1.$$

9. [Dual variational problems] Let V be a finite dimensional subspace of a Hilbert Space H and W its orthogonal complement. Recall that we can decompose any $x \in H$ uniquely as

$$x = x_V + x_W, \qquad \text{where } x_V \in V, \text{ and } x_W \in W.$$

a) Show that $\max\limits_{\{z \perp V, \|z\|=1\}} \langle x, z \rangle = \|x_W\|$.

b) Show that $\min\limits_{v \in V} \|x - v\| = \|x_W\|$.

[Remark: *dual variational problems* are a pair of maximum and minimum problems whose extremal values are equal.]

Bonus Problem 7. Say one has a real normed linear space whose norm satisfies the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Define the bilinear form $B(x,y)$ by the rule (see Problem 4 above)

$$B(x,y) := \tfrac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right].$$

a) Show that $B(x,y)$ has all of the properties of an inner product including $B(x,x) = \|x\|^2$. [First observed by von Neumann]

MORAL: If a norm satisfies the parallelogram identity, then one can use it to define a compatible inner product; the parallelogram identity is both necessary and sufficient for the norm to arise from an inner product.

b) Consider $\mathbb{R}^2$ with vectors $X = (x_1, x_2)$ having the p norm, $\|x\|_p := (|x_1|^p + |x_2|^p)^{1/p}$, $p \geq 1$. Use the result of the previous problem to how that if $p \neq 2$, this norm does *not* arise from an inner product.

[Last revised:March 12, 2007]