Math 509: Problem Set 5 (due Thurs. Feb 15, 2007)

1. This concerns the curve $y = |x|$ for $-1 \le x \le 1$.
   a) Find a parameterization $x = \phi(t)$, $y = \psi(t)$, where $\phi(t)$ and $\psi(t)$ are both functions in $C^4([-1,1]]$ with $\phi(t)$ an increasing function of t.
   b) Repeat this using functions $\phi, \psi \in C^\infty([-1,1]]$.

2. [Rudin, p. 165 #1] Prove that every uniformly convergent sequence of bounded functions $f_n(x)$ is uniformly bounded.

3. [Rudin, p. 165 #2] If $\{f_n(x)\}$ and $\{g_n(x)\}$ are sequences of bounded functions that converge uniformly for x in a set E prove that $f_n g_n$ also converges uniformly on E.

4. a) In a complete metric space M with distance $d(x,y)$, let $x_j \in M$ be a sequence that satisfies $\sum_j d(x_{j+1}, x_j) < \infty$. Show that the x_j converge to an element of M.
   b) Give an example showing that if $d(x_{j+1}, x_j) \to 0$, then the sequence x_j might not converge.
   c) Let $\|f\| = \max_{0 \le x \le 2} |f(x)|$ for $f \in C([0,2])$. If $f_j \in C([0,2])$ $j = 1, 2, \dots$ satisfies
   $$\|f_{j+1} - f_j\| \le \frac{1}{j^2},$$
   show that the f_j converge uniformly in the interval $[0,2]$.

5. Let $\varphi(x)$, $x \in \mathbb{R}^n$ be a smooth function with the following properties
   i). $\varphi(x) > 0$ for $\|x\| < 1$, $\varphi(x) = 0$ for $\|x\| \ge 1$,
   ii). $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$.

   Let $\varphi_k(x) := k^n \varphi(kx)$. For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set $\mathcal{K}$, define
   $$f_k(x) := \int_{\mathbb{R}^n} f(t)\varphi_k(x-t)\, dt.$$
   a) Give an example of a function φ with these properties.
   b) Show that $\varphi_k(x) = 0$ for $\|x\| \ge 1/k$, and $\int_{\mathbb{R}^n} \varphi_k(x)\, dx = 1$.
   c) Show that the f_k are smooth functions.
   d) Show that $\lim_{k\to\infty} f_k(x) = f(x)$, and that this convergence is uniform.

6. Given continuous function $h(x,y)$ and $f(x)$ for all real x, y. For some constant c and $0 \le x \le c$, we seek a solution $u(x)$ of the *integral equation*

$$u(x) = f(x) + \int_0^c h(x,y)u(y)\,dy. \tag{*}$$

as follows. Let $u_0(x) \equiv 0$ and define $u_k(x)$, $k = 1,2,\ldots$, recursively by the rule

$$u_{k+1}(x) = f(x) + \int_0^c h(x,y)u_k(y)\,dy.$$

a) Show that if $c > 0$ is sufficiently small, then the $u_k(x)$ converge uniformly for $0 \le x \le c$ to a continuous function $u(x)$ that satisfies the integral equation (*).

b) In the special case where $h(x,y) := \sum_{i=1}^{N} a_i(x)b_i(y)$ (where the functions a_i and b_i are, say, continuous), then equation (*) can be written as

$$u(x) = f(x) + \sum_{i=1}^{N} Q_i a_i(x), \quad \text{where} \quad Q_i := \int_0^c b_i(y)u(y)\,dy. \tag{**}$$

With this observation, show that one can reduce (*) to a system of N linear algebraic equations:

$$Q_i = \gamma_i + \sum_{j=1}^{N} \alpha_{ij} Q_j,$$

where

$$\gamma_i := \int_0^c b_i(x) f(x)\,dx \quad \text{and} \quad \alpha_{ij} := \int_0^c b_i(x)a_j(x)\,dx.$$

Thus the γ_i and α_{ij} are regarded as known constants and the Q_i are the unknowns. [Suggestion: In (**) substitute the formula for u back into the formula for Q_i.]

c) In the special case where $h(x,y) \equiv 1$ and $f(x) \equiv 1$, solve equation (*) explicitly. From this, show that indeed for some value of c a solution may *not* exist.

Bopnus Problem 5-A Let $f(x)$ be a continuous real-valued function with period 2π, so $f(x+2\pi) = f(x)$ for all real x. If also for some *irrational* $\alpha \in \mathbb{R}$ we know that $f(x+2\pi\alpha) = f(x)$ for all real x, show that $f(x) \equiv \text{constant}$.

Bonus Problem 5-B Let α be an irrational real number and let $f(\theta)$ be a continuous 2π periodic function, $0 \le \theta \le 2\pi$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(2\pi k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\,d\theta\,.$$

[Last revised:February 15, 2007]