Math 509: Problem Set 3 (due Tues. Jan. 30, 2007)

1. A standard ingredient in many problems involves the eigenvalues λ and corresponding eigenfunctions u of the Laplacian $\Delta = \nabla\cdot\nabla$, so

$$-\Delta u = \lambda u \quad \text{in} \quad \mathcal{D} \qquad \text{with} \qquad u = 0 \quad \text{on} \quad \mathcal{B},$$

Here $\mathcal{D}$ in $\mathbb{R}^2$ is a bounded region with boundary $\mathcal{B}$. As usual, to be useful one wants numbers λ so that there is a solution u other than the trivial solution $u \equiv 0$. Show that

$$\lambda = \frac{\iint_{\mathcal{D}} |\nabla u|^2\, dA}{\iint_{\mathcal{D}} u^2\, dA}.$$

In particular, deduce that $\lambda > 0$.

2. a) Show that for any smooth function $u(x,y)$

$$\iint_{\mathcal{D}} \Delta\varphi\, dx\, dy = \int_{\partial\mathcal{D}} \frac{\partial\varphi}{\partial N}\, ds$$

where $\partial\varphi/\partial N := \nabla\varphi\cdot\mathbf{N}$ is the outer normal directional derivative on $\partial\mathcal{D}$.

b) Let $u(x,y,t)$ be a solution of the *heat equation* $u_t = \Delta u$ for (x,y) in $\mathcal{D}$. Assume that the boundary, $\partial\mathcal{D}$, is *insulated*, so the outer normal derivative there is zero: $\frac{\partial u}{\partial N} = 0$ on $\partial\mathcal{D}$.

Show that $\quad Q(t) := \iint_{\mathcal{D}} u(x,y,t)\, dx\, dy \quad$ is a constant.

3. a) If the sequence $\{a_n\}$ is bounded and $c > 1$, show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges absolutely and uniformly for all complex $z = x + iy$ in the closed half-plane $c \le x < \infty$.

b) For real x consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^x}$. Describe the values of x where this converges absolutely. Describe the sets of x where this converges uniformly.

4. For which subsets of $\mathbf{R}$ does the series

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$$

converge uniformly? (*Hint*: Sum the series!)

5. (Rudin, p. 166 #6) Prove that the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in any bounded interval – but does not converge absolutely for *any* value of x.

6. (Rudin, p. 166 #7) For $n = 1, 2, \ldots$ and $x \in \mathbb{R}$, define $f_n(x) := \dfrac{x}{1+nx^2}$.

   a) Show that f_n converges uniformly to a function f.

   b) Show that $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$ but false if $x = 0$.

7. Say one has a continuous function $f(x)$ defined for *all* real x with the property that there is a sequence of polynomials $p_k(x)$ that converge uniformly to f for all x. Thus, given any $\varepsilon > 0$, then for all sufficiently large k we have

$$\sup_{x\in\mathbb{R}} |f(x) - p_k(x)| < \varepsilon.$$

   Show that $f(x)$ must itself be a polynomial.

8. a) Let $\{a_n\}$ be a sequence of real numbers with the property that

$$|a_{k+1} - a_k| \le \tfrac{1}{2}|a_k - a_{k-1}|, \qquad k = 1, 2, \ldots .$$

   Show that this sequence converges to some real number.

   b) NOTATION: $\|\varphi\| = \sup_{0\le x\le 1} |\varphi(x)|$ (so this is the uniform norm). Using this notation, let $\{u_n(x)\}$ be a sequence of continuous functions for $0 \le x \le 1$ with the property that

$$\|u_{k+1} - u_k\| \le \tfrac{1}{2}\|u_k - u_{k-1}\|, \qquad k = 1, 2, \ldots.$$

   Show that the $\{u_n\}$ converge uniformly to a continuous function.

[Last revised:January 21, 2007]