Math 509: Problem Set 2 (due Tues. Jan. 23, 2007)

1. Let $A \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$.

   a) Find a symmetric matrix P so that $P^2 = A$ (so P is the square root of A).

   b) Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and let A be an invertible symmetric matrix, $b \in \mathbb{R}^n$ be a vector and $c \in \mathbb{R}$ a scalar. Define the quadratic polynomial

   $$Q(x) = \langle x, Ax \rangle + \langle b, x \rangle + c.$$

   Find a vector $v \in \mathbb{R}^n$ so that the after the change of variables $x = y - v$ (a translation) then Q has the simpler form $Q = \langle y, Ay \rangle + \gamma$, for some scalar γ. [This is a generalization of *completing the square* from high school algebra].

   c) Find new coordinates in which the quadratic polynomial $5x^2 - 4xy + 5y^2 - 2x + 3$ has the simpler form $u^2 + v^2 + c$ for some constant c.

2. Let A be a positive definite $n \times n$ symmetric matrix, x and b vectors in $\mathbb{R}^n$. For which value(s) of the real number c is the "plane" $\{x \in \mathbb{R}^n \mid \mid \langle b, x \rangle = c\}$ tangent to the quadratic "surface" $\{x \in \mathbb{R}^n \mid \langle x, Ax \rangle = 1\}$? [SUGGESTION: this is easy if A is the identity matrix.]

3. Let A be an $n \times n$ symmetric matrix and $b \in \mathbb{R}^n$ a given vector. Show that if $x \in \mathbb{R}^n$ is a critical point of the function

   $$f(x) := \tfrac{1}{2}\langle x, Ax \rangle - < b, x >,$$

   then it is a solution of $Ax = b$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. If $f(0) = 0$ and $\nabla f(0) = 0$, show there is a symmetric matrix $H(x)$ whose elements are smooth functions, with $f(x) = \langle x, H(x)x \rangle$. [SUGGESTION: first do this for $n = 1$].

5. If $p > 1$ and $1/p + 1/q = 1$, recall that in Homework 1 you proved the inequality $uv \le u^p/p + v^q/q$ for all $u \ge 0$ and $v \ge 0$.

   a) If $f(x) \ge 0$ and $g(x) \ge 0$ are continuous functions with $\int_a^b f(x)^p \, dx = 1$ and $\int_a^b g(x)^q \, dx = 1$, show that $\int_a^b f(x)g(x) \, dx \le 1$.

b) [Hölder] More generally, If $f(x)$ and $g(x)$ are continuous functions, show that

$$\int_a^b |f(x)g(x)|\,dx \le \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \left[\int_a^b |g(x)|^q\,dx\right]^{1/q}.$$

[SUGGESTION: in one line reduce this to the previous part.]

c) Define $\|f\|_p$ as:

$$\|f\|_p = \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}.$$

Use Hölder's inequality above to prove the *triangle inequality*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

6. [cf Rudin, p. 239 #12] If $0 < a < b$ and Q is the square $0 \le s < 2\pi$, $0 \le t < 2\pi$ in $\mathbb{R}^2$, show that the following map from Q to $\mathbb{R}^3$ describes the surface of a torus:

$$\begin{aligned} x(s,t) &= (b + a\cos s)\cos t \\ y(s,t) &= (b + a\cos s)\sin t \\ z(s,t) &= a\sin s \end{aligned}$$

Also, find and classify the critical points of $x(s,t)$.

7. (Euler) Let $f(x)$ be a differentiable function of $x \in \mathbb{R}^n$. If f is *homogeneous of degree k* in the sense that $f(cx) = c^k f(x)$ for all $c > 0$, show that $x \cdot \nabla f(x) = kf(x)$.

8. Let $B \subset \mathbb{R}^2$ be the rectangle $0 \le x \le 2$, $0 \le y \le 1$. Show that

$$\iint_B \left[\frac{x^2 + 2y^3}{2 + \sin(xy)}\right]^{1/2} dx\,dy \le 2 + \sqrt{7}.$$

[SUGGESTION: Split B into two smaller regions.]

9. Let $f(x) = x\sin(\pi/x)$ for $x \ne 0$ and $f(0) = 0$. Draw a sketch of the curve $y = f(x)$ for $0 \le x \le \pi$. Using straight line segment approximations to the arc length, show that this curve has infinite are length. [As a first step, show geometrically that the length of the portion (one arch) of the curve for $\frac{1}{n+1} \le x \le \frac{1}{n}$ is at least $2/(n + \frac{1}{2})$.]

This is the standard example of a non-rectifiable curve.

10. a) Compute

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+4x^2+9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\,dy}{(5+x^2+2x+9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+5x^2-4xy+5y^2)^2}.$$

b) Let $h(t)$ be a given function and say you know that $\int_0^\infty h(t)\,dt = \alpha$. If C be a positive definite 2×2 matrix. Show that

$$\iint_{\mathbb{R}^2} h(\,\langle x, Cx\rangle\,)\;dA = \frac{\pi\alpha}{\sqrt{\det C}}.$$

c) Compute $\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2)}\,dx\,dy$.

d) Compute $\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2-2x+3)}\,dx\,dy$.

e) Generalize part b) to obtain a formula for

$$\iint_{\mathbb{R}^n} h(\,\langle x, Cx\rangle\,)\;dV,$$

where now C be a positive definite $n\times n$ matrix. The answer will involve some integral involving h and also the "area" of the unit sphere $S^{n-1} \hookrightarrow \mathbb{R}^n$.

11. Let $f(x)$ be a continuous function for $0 \le x \le 1$. Evaluate $\lim_{n\to\infty} n\int_0^1 f(x)x^n\,dx$. (Justify your assertions.)

Bonus Problem Let $\gamma(t)$ be a closed piecewise smooth curve in the plane that encloses a convex region D. We say that a curve γ_r is *parallel* to γ at distance r if every point on γ_r is outside D and has distance r from γ. For instance, concentric circles are parallel. Discover a formula relating the arc length of γ_r to γ. [SUGGESTION: One approach begins by looking at the cases where γ is a circle, a rectangle, and a (convex) polygon.]

[Last revised:January 21, 2007]