Math 509: Problem Set 10 (due Thurs. April 12, 2007)

1. Let $u(\theta,t)$ be the temperature at a point θ on the circle, $S^1 = [-\pi, \pi]$ at time t and assume that $u(\theta,t)$ satisfies the *heat equation*
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}$$
and is of course periodic in θ with period 2π. If the initial temperature is $u(\theta,0) = f(\theta) \in C(S^1)$, show that
$$\lim_{t\to\infty} u(\theta,t) = \text{constant}$$
and determine this constant in terms of f.

2. Solve the Laplace equation $\Delta u = 0$ on the *outside* of the unit disk, so $r = \sqrt{x^2+y^2} > 1$ with $u(r,\theta)|_{r=1} = 2+\cos\theta - 3\sin\theta$ on the unit circle (here θ is the angle in polar coordinates). In you solution, assume that $u(r,\theta)$ is bounded as $r \to \infty$.

3. Suppose that u is a harmonic function in the disk $D = \{x^2+y^2 < 4\}$ with boundary condition (in polar coordinates) $u(2,\theta) = 5+2\cos 3\theta$. Without computing the solution:
   a) Find the maximum value of the solution in $\overline{D}$.
   b) Find the minimum value of the solution in $\overline{D}$.
   c) Find the value of u at the origin.

4. Let u and v be harmonic functions in a bounded (connected) region D with $u = f$ and $v = g$ on the boundary of D. If $f \le g$, show that $u \le v$ with strict inequality everywhere unless $f \equiv g$,

5. Say $\Delta u \ge 0$ in a bounded (connected) region D.
   a) Prove that in any disk $Q \in D$ the value of u at the center of the disk is at most the average of its value on the boundary of the disk.
   b) Show that if u takes its maximum value at an interior point of D then u must be a constant.
   c) Assume $\Delta v = F$ and $\Delta w = G$ in a bounded (connected) region D with $v = 0$ and $w = 0$ on the boundary of D. If $F > G$ in D, show that $v < w$ in D.