# Advanced Analysis: Problem Set 1 (due Tues. Jan 16)

**Math 509, Spring 2007** **Jerry L. Kazdan**

*This first problem set should all be review of material you already know.*

1. In a metric space, if a subsequence of a Cauchy sequence $x_j$ converges to some point $x$, show that the whole sequence also converges to $x$.

2. Let $p > 1$ and define $q$ by $1/p + 1/q = 1$. Prove **Young's inequality**: for all real $r$ and $s$:
$$|rs| \le \frac{|r|^p}{p} + \frac{|s|^q}{q}.$$
[REMARK: There are many (elementary) ways to show this; none of them are completely obvious.]

3. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 < p < \infty$, let $\|x\|_p := \left[\sum |x_j|^p\right]^{1/p}$.
   a) As a corollary of the previous problem prove **Hölder's inequality**:
$$\sum_{j=1}^{n} |x_j y_j| \le \|x\|_p \|y\|_q$$
   b) Use Hölder's Inequality (see above) to prove the triangle inequality for the $p$-norm: $\| \ \|_p$.

4. A set $D$ in the plane is *arcwise connected* if for any two points in $D$ there is a polygonal curve (in $D$) joining them. Show that an *open* connected set is arcwise connected.

5. Compute $\int_0^a x^2\,dx$ directly by using Riemann sums (not as the anti-derivative). You may use without proof that $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$.

6. Find a smooth function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ that has exactly three critical points in $\mathbb{R}^2$, one a local max, one a local min, and one a saddle point. [SUGGESTION: Seek $f$ in the special form $f(x,y) = g(x)/(1+y^2)$.]

7. Find all of the critical points of $f(x,y) = (x^2 + 2y^2)e^{-(x^2+y^2)}$. Here $(x,y)$ is a point in $\mathbb{R}^2$. Classify these points as max, min, or saddles. You may find it interesting to plot this using Maple. Here is the code:

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with(plots):
f:=(x,y) ->(x^2+2*y^2)*exp(-(x^2+y^2));
plot3d(f(x,y), x=-6..6, y=-3..3, orientation=[20,70], grid=[60,60]);
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8. Let $u(x)$ be a smooth function of $x = (x_1, \ldots, x_n)$ and assume that the Hessian matrix

$$u''(p) := \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)$$

at the point $p = (p_1, \ldots, p_n)$ is non-singular (that is, it is invertible).

a) At a point $p$, let $v \neq 0$ be a given vector, $\varphi(t) := u(p+tv)$, $t \in \mathbb{R}$, be the restriction of $u$ to the straight line $x = p+tv$. Show that the Hessian matrix $u''(p)$ is positive definite if and only if $\varphi''(0) > 0$ for all directions $v$. Is it enough if we only check the vectors $v$ in an orthonormal basis?

b) Use the example $u(x,y) = (y-x^2)(y-2x^2)$ to show the conclusion may be false if $u''(p)$ is not invertible.

9. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and assume that $u(x)$ depends only on the distance to the origin, $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, so $u(x) = f(r)$ for some function $f$ depending only on $r$.

a) Show that $\dfrac{\partial u}{\partial x_i} = \dfrac{x_i}{r} \dfrac{df}{dr}$.

b) Show that $\dfrac{\partial^2 u}{\partial x_i^2} = \dfrac{d^2 f}{dr^2} \left( \dfrac{x_i^2}{r^2} \right) + \dfrac{df}{dr} \left( \dfrac{1}{r} - \dfrac{x_i^2}{r^3} \right)$.

c) Compute $\Delta u := \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \ldots + \dfrac{\partial^2 u}{\partial x_n^2}$ in terms of $f$ and its derivatives.

d) If $n = 3$, use this to find all functions $u(x) = f(r)$ that satisfy $\Delta u = 0$ for all $x \neq 0$.

**Bonus Problem:** Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with exactly one critical point, and that critical point is a strict local minimum (say the critical, point is at the origin and the Hessian matrix there is the identity matrix, $f''(0,0) = I$). Can one conclude that the origin is the *global* minimum? Proof or counter-example.