

## The Laplace Equation on a Disk

TO SOLVE:  $\Delta u = 0$  in a disk of radius  $a$  with the Dirichlet boundary conditions  $u = h$  on the boundary  $x^2 + y^2 = a^2$ .

In polar coordinates  $u(r, \theta)$ :

$$u_{rr} + \frac{1}{r}u_r + u_{\theta\theta} = 0 \quad \text{with} \quad u(a, \theta) = h(\theta).$$

Using separation of variables, seek special solutions  $u(r, \theta) = R(r)W(\theta)$ , where  $R(r)$  is smooth for  $0 \leq r < 1$  and  $W(\theta)$  is periodic with period  $2\pi$ . This leads to the special solutions

$$u_k(r, \theta) = r^k(A_k \cos k\theta + B_k \sin k\theta)$$

so we seek the “general” solution of  $\Delta u = 0$  in the disk in the form

$$u(r, \theta) = \sum_{k=0}^{\infty} r^k(A_k \cos k\theta + B_k \sin k\theta).$$

To satisfy the boundary condition  $u(a, \theta) = h(\theta)$  we need

$$h(\theta) = \sum_{k=0}^{\infty} a^k(A_k \cos k\theta + B_k \sin k\theta)$$

where the coefficients  $a_k$  and  $b_k$  are found using the orthogonality of  $\cos k\theta$  and  $\sin k\theta$ :

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) d\phi$$

$$A_k = \frac{1}{\pi a^k} \int_{-\pi}^{\pi} h(\phi) \cos k\phi d\phi \quad (k \geq 1)$$

$$B_k = \frac{1}{\pi a^k} \int_{-\pi}^{\pi} h(\phi) \sin k\phi d\phi.$$

Substituting these into the formula for  $u(r, \theta)$  we get

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) d\phi$$

$$+ \frac{1}{\pi} \sum_1^{\infty} \left(\frac{r}{a}\right)^k \int_{-\pi}^{\pi} h(\phi) [\cos k\phi \cos k\theta + \sin k\phi \sin k\theta] d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) \left[ 1 + 2 \sum_1^{\infty} \left(\frac{r}{a}\right)^k \cos(\theta - \phi) \right] d\phi.$$

But the above sum gives geometric series:

$$P(r, \theta) := 1 + 2 \sum_1^{\infty} \left(\frac{r}{a}\right)^k \cos \theta = 1 + \sum_1^{\infty} \left(\frac{r}{a}\right)^k (e^{ik\theta} + e^{-ik\theta})$$

$$= \frac{a^2 - r^2}{a^2 - 2ar \cos \theta + r^2}.$$

so we obtain the solution as a convolution:

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) P(r, \theta - \phi) d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta - \phi) P(r, \phi) d\phi. \end{aligned}$$

The function  $P(r, \theta)$  is called the *Poisson kernel*. It has the properties

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta) d\theta = 1 \quad \text{and} \quad P(r, \theta) > 0 \quad \text{for} \quad r < a.$$

Also, if we let  $p = (r, \theta)$  be a point in the disk and  $q = (a, \phi)$  a point on the boundary, then by the law of cosines,  $a^2 - 2ar \cos(\theta - \phi) + r^2 = |p - q|^2$ .

CLAIM: If  $h \in C(S^1)$ , then  $u(r, \theta)$  converges uniformly to  $h(\theta)$  uniformly as  $r \nearrow 1$ .

PROOF: This is very similar to many proofs we have seen this semester. Since  $h$  is uniformly continuous on  $S^1$ , then given any  $\varepsilon > 0$  there is a  $\delta > 0$  so that if  $|\theta_2 - \theta_1| < \delta$  then  $|h(\theta_2) - h(\theta_1)| < \varepsilon$ . Also for some constant  $M$  we have  $|h(\theta)| \leq M$  on  $S^1$ . Thus

$$\begin{aligned} u(r, \theta) - h(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(\theta - \phi) - h(\theta)] P(r, \phi) d\phi \\ &= J_1 + J_2, \end{aligned}$$

where

$$|J_1| = \left| \frac{1}{2\pi} \int_{|\phi| < \delta} [h(\theta - \phi) - h(\theta)] P(r, \phi) d\phi \right| \leq \frac{\varepsilon}{2\pi} \int_{|\phi| < \delta} P(r, \phi) \leq \varepsilon$$

and, since  $|h(\theta)| \leq M$ ,

$$\begin{aligned} |J_2| &= \left| \frac{1}{2\pi} \int_{\delta \leq |\phi| \leq \pi} [h(\theta - \phi) - h(\theta)] P(r, \phi) d\phi \right| \\ &\leq \frac{2M(a^2 - r^2)}{2\pi} \int_{\delta \leq |\phi| \leq \pi} \frac{1}{a^2 - 2ar \cos \phi + r^2} d\phi \\ &\leq \frac{2M(a^2 - r^2)}{a^2 - 2ar \cos \delta + r^2} \end{aligned}$$

As  $r \nearrow a$ , this last term goes to zero. Since these estimates for  $J_1$  and  $J_2$  are independent of  $\theta$ , the convergence is uniform for  $\theta \in S^1$ .