Math 509, Spring 2007 Jerry L. Kazdan

The Laplace Equation on a Disk

TO SOLVE: $\Delta u = 0$ in a disk of radius a with the Dirichlet boundary conditions $u = h$ on the boundary $x^2 + y^2 = a^2$.
In polar coordinates $u(r,\theta)$:

$$u_{rr} + \frac{1}{r}u_r + u_{\theta\theta} = 0 \qquad \text{with} \qquad u(a,\theta) = h(\theta).$$

Using separation of variables, seek special solutions $u(r,\theta) = R(r)W(\theta)$, where $R(r)$ is smooth for $0 \le r < 1$ and $W(\theta)$ is periodic with period 2π. This leads to the special solutions

$$u_k(r,\theta) = r^k(A_k \cos k\theta + B_k \sin k\theta)$$

so we seek the "general" solution of $\Delta u = 0$ in the disk in the form

$$u(r,\theta) = \sum_{k=0}^{\infty} r^k(A_k \cos k\theta + B_k \sin k\theta).$$

To satisfy the boundary condition $u(a,\theta) = h(\theta)$ we need

$$h(\theta) = \sum_{k=0}^{\infty} a^k(A_k \cos k\theta + B_k \sin k\theta)$$

where the coefficients a_k and b_k are found using the orthogonality of $\cos k\theta$ and $\sin k\theta$:

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,d\phi \\
A_k &= \frac{1}{\pi a^k}\int_{-\pi}^{\pi} h(\phi)\cos k\phi\,d\phi \quad (k \ge 1) \\
B_k &= \frac{1}{\pi a^k}\int_{-\pi}^{\pi} h(\phi)\sin k\phi\,d\phi.
\end{aligned}$$

Substituting these into the formula for $u(r,\theta)$ we get

$$\begin{aligned}
u(r,\theta) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,d\phi \\
&+ \frac{1}{\pi}\sum_{1}^{\infty}\left(\frac{r}{a}\right)^k \int_{-\pi}^{\pi} h(\phi)[\cos k\phi \cos k\theta + \sin k\phi \sin k\theta]\,d\phi \\
=& \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\left[1 + 2\sum_{1}^{\infty}\left(\frac{r}{a}\right)^k \cos(\theta - \phi)\right]d\phi.
\end{aligned}$$

But the above sum gives geometric series:

$$\begin{aligned} P(r,\theta) :=& 1+2\sum_1^\infty \left(\frac{r}{a}\right)^k \cos\theta = 1+\sum_1^\infty \left(\frac{r}{a}\right)^k (e^{ik\theta}+e^{-ik\theta}) \\ =& \frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}. \end{aligned}$$

so we obtain the solution as a convolution:

$$\begin{aligned} u(r,\theta) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)P(r,\theta-\phi)\,d\phi \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta-\phi)P(r,\phi)\,d\phi. \end{aligned}$$

The function $P(r,\theta)$ is called the *Poisson kernel*. It has the properties

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 1 \quad \text{and} \quad P(r,\theta)>0 \quad \text{for} \quad r<a.$$

Also, if we let $p=(r,\theta)$ be a point in the disk and $q=(a,\phi)$ a point on the boundary, then by the law of cosines, $a^2-2ar\cos(\theta-\phi)+r^2 = |p-q|^2$.

CLAIM: If $h\in C(S^1)$, then $u(r,\theta)$ converges uniformly to $h(\theta)$ uniformly as $r\nearrow 1$.

PROOF: This is very similar to many proofs we have seen this semester. Since h is uniformly continuous on S^1, then given any $\varepsilon>0$ there is a $\delta>0$ so that if $|\theta_2-\theta_1|<\delta$ then $|h(\theta_2)-h(\theta_1)|<\varepsilon$. Also for some constant M we have $|h(\theta)|\le M$ on S^1. Thus

$$\begin{aligned} u(r,\theta)-h(\theta) =& \frac{1}{2\pi}\int_{-\pi}^{\pi}[h(\theta-\phi)-h(\theta)]P(r,\phi)\,d\phi \\ =& J_1+J_2, \end{aligned}$$

where

$$|J_1| = \left|\frac{1}{2\pi}\int_{|\phi|<\delta}[h(\theta-\phi)-h(\theta)]P(r,\phi)\,d\phi\right| \le \frac{\varepsilon}{2\pi}\int_{|\phi|<\delta} P(r,\phi) \le \varepsilon,$$

and, since $|h(\theta)|\le M$,

$$\begin{aligned} |J_2| =& \left|\frac{1}{2\pi}\int_{\delta\le|\phi|\le\pi}[h(\theta-\phi)-h(\theta)]P(r,\phi)\,d\phi\right| \\ \le& \frac{2M(a^2-r^2)}{2\pi}\int_{\delta\le|\phi|\le\pi}\frac{1}{a^2-2ar\cos\phi+r^2}\,d\phi \\ \le& \frac{2M(a^2-r^2)}{a^2-2ar\cos\delta+r^2} \end{aligned}$$

As $r\nearrow a$, this last term goes to zero. Since these estimates for J_1 and J_2 are independent of θ, the convergence is uniform for $\theta\in S^1$.