Math 509 — May 1, 2007

Final Exam

Jerry L. Kazdan — 9:00 –11:00

DIRECTIONS This exam has three parts, Part A has 4 shorter problems (5 points each), Part B has 5 traditional problems (10 points each).
Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Shorter Problems (4 problems, 5 points each).

A–1. Give an example of a sequence of continuous functions $f_n(x)$, $0 \le x \le 1$, with $f_n(x) \to 0$ (pointwise) for all $x \in [0,1]$, but $\int_0^1 |f_n(x)|\,dx \to \infty$. A sketch is adequate.

A–2. In $L_2(-1,1)$ with the standard inner product, show that any even function is orthogonal to any odd function (of course assume that the functions are integrable).

A–3. Prove that the series $\displaystyle\sum_1^\infty \frac{(-1)^k \sin kx}{1+k^2}$ converges absolutely and uniformly for all real x.

A–4. Let $u(x,y,t)$ be a solution of the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ for $(x,\,y)$ in a bounded domain $D \in \mathbb{R}^2$ with the outer normal derivative $\nabla u \cdot N = 0$ on the boundary of D (here N is the unit outer normal vector field on the boundary).

If $Q(t) := \displaystyle\iint_D u(x,y,t)\,dx\,dy$, show that $\dfrac{dQ}{dt} = 0$ and hence that $Q(t) = Q(0)$.

Part B: Traditional Problems (5 problems, 10 points each)

B–1. The following equations define a map $F : (x,y,z) \mapsto (u,v,w)$:

$$\begin{aligned} u(x,y,z) &= x + xyz^2 \\ v(x,y,z) &= xz^2 + y \\ w(x,y,z) &= 2x + cz + z^3 \end{aligned}$$

Clearly $F : (1,1,0) \mapsto (1,1,2)$. Write $p = (1,1,0)$ and $q = (1,1,2)$.

a) Compute the derivative $F'(p)$.

b) For which value(s) of the constant c can the system of equations: can be solved for x, y, z as smooth functions of u, v, w near p? Justify your assertion(s).

c) If c is one of these "good" values, let $G : (u,v,w) \mapsto (x,y,z)$ be the map inverse to F. Compute the derivative $G'(q)$ and use it to compute $\partial y(u,v,w)/\partial v$ at q.

B–2. In a Hilbert space $\mathcal{H}$, let $v_1, \dots v_n$ be orthonormal vectors and $x \in \mathcal{H}$ a given vector.

a) Show there are scalars $a_1, \dots, a_n$ and a $w \in \mathcal{H}$ with $w \perp \{v_1, \dots, v_n\}$ so that

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + w.$$

Your work should exhibit a formula for the a_k in terms of x and $v_1, \dots, v_n$.

b) Show that $\|x\|^2 = |a_1|^2 + \cdots + |a_n|^2 + \|w\|^2$.

B–3. Let u and v be harmonic functions in a bounded (connected) region D with $u = f$ and $v = g$ on the boundary of D. If $f < g$, show that $u < v$.

B–4. In homework you found that the Fourier series for $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ -1 & \text{for } -\pi \le x < 0. \end{cases}$ is

$$f(x) \sim \frac{2}{i\pi}\left[\left(\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots\right) - \left(\frac{e^{-ix}}{1} + \frac{e^{-3ix}}{3} + \frac{e^{-5ix}}{5} + \cdots\right)\right].$$

Use this and the Parseval Theorem to compute

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

B–5. Let $\varphi_k(x)$, $x \in \mathbb{R}$, be a sequence of smooth functions with the following properties

i). $\varphi_k(x) \ge 0$ for $|x| < 1/k$, $\quad \varphi_k(x) = 0$ for $|x| \ge 1/k$,

ii). $\int_{\mathbb{R}} \varphi_k(x)\, dx = 1$.

For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set $\mathcal{K}$, define

$$f_k(x) := \int_{\mathbb{R}} f(y)\varphi_k(x - y)\, dy.$$

Show that $\lim_{n\to\infty} f_k(x) = f(x)$, and that this convergence is uniform.