Math 509 | **Exam 1** | Jerry Kazdan
March 1, 2007 | | 12:00 — 1:20

DIRECTIONS This exam has two parts, Part A has 2 short answer problems (5 points each) while Part B has 5 traditional problems (10 points each). Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Proof or Counterexample (2 problems, 5 points each)
Here let $f_n(x)$, $n = 1, 2, \ldots$ be a sequence of continuous functions for $0 \le x \le 2$. *For a counterexample, a clear sketch may be completely adequate.*

A–1. If $f_n(x)$ converges to zero for *every* $x \in [0, 2]$, then f_n converges to zero *uniformly* on the interval $[0, 2]$.

A–2. If $f_n(x)$ converges uniformly to zero for x in the interval $[0, 2]$, then $\int_0^2 f_n(x)\,dx \to 0$.

Part B: Traditional Problems (5 problems, 10 points each)

B–1. Compute $\int_0^a x^2\,dx$ (where $0 < a < \infty$) directly by using Riemann sums (not as the anti-derivative). I suggest partitioning the interval $0 \le x \le a$ into segments having equal length. You may use without proof that $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$.

B–2. Let a_n be a *bounded* sequence of real numbers. If $c > 1$, show that the series $\sum_1^\infty \frac{a_n}{n^x}$ converges uniformly for $x \ge c$.

B–3. Let $f(x) \in C([0, 1])$ be a continuous function with the property: $\int_0^1 f(x)p(x)dx = 0$ for *every* polynomial $p(x)$. Show that $f(x) \equiv 0$.

B–4. Let

$$p(x) := (x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = x^6 - 21x^5 + \cdots$$

Clearly $p(4) = 0$. Denote by $p(x, t)$ the polynomial obtained by replacing $-21x^5$ by $-(21+t)x^5$, with $|t|$ small. Let $x(t)$ denote the perturbed value of root $x = 4$, so $x(0) = 4$.

a) Show that $x(t)$ is a smooth function of t for all $|t|$ sufficiently small.

b) Compute the sensitivity of this root as one changes t, that is, compute $dx(t)/dt\,\big|_{t=0}$.

B–5. Let $f(x)$ and $h(x, y)$ be continuous for x and y in the interval $[0, 2]$. Show that if $\lambda > 0$ is sufficiently small, the equation

$$u(x) = f(x) + \lambda \int_0^2 h(x, y)u(y)\,dy$$

has a unique solution (that is, a solution exists and is unique).