Math 509 **Exam 1** Jerry Kazdan
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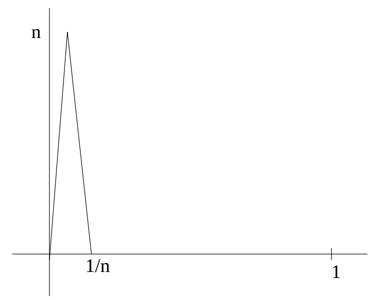
DIRECTIONS This exam has two parts, Part A has 2 short answer problems (5 points each) while Part B has 5 traditional problems (10 points each). Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Proof or Counterexample (2 problems, 5 points each)
Here let $f_n(x)$, $n = 1, 2, \ldots$ be a sequence of continuous functions for $0 \le x \le 2$. *For a counterexample, a clear sketch may be completely adequate.*

A–1. If $f_n(x)$ converges to zero for *every* $x \in [0, 2]$, then f_n converges to zero *uniformly* on the interval $[0, 2]$.

Solution: No. See the figure.



A–2. If $f_n(x)$ converges uniformly to zero for x in the interval $[0, 2]$, then $\int_0^2 f_n(x)\,dx \to 0$.

Solution: Yes. Given $\epsilon > 0$, pick N so that if $n > N$ then $|f_n(x)| < \epsilon$ for all $x \in [0, 2]$. Then

$$\left| \int_0^2 f_n(x)\,dx \right| \le \int_0^2 |f_n(x)|\,dx < 2\epsilon.$$

Part B: Traditional Problems (5 problems, 10 points each)

B–1. Compute $\displaystyle\int_0^a x^2\,dx$ (where $0 < a < \infty$) directly by using Riemann sums (not as the anti-derivative). I suggest partitioning the interval $0 \le x \le a$ into segments having equal length. You may use without proof that $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$.

Solution: Since the function $f(x) = x^2$ is continuous and $[0, a]$ is compact, we know its Riemann integral exists so we can evaluate it using any convenient partition $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = a$ of the interval. We use a partition having N segments, each of length $\Delta x_j = a/N$ and evaluate the function at the right end of each segment. Then $x_j = aj/N$, $j = 0, .., N$. Consequently

$$\int_0^a x^2\,dx \approx \sum_{j=1}^N x_j^2 \Delta x_j = \sum_{j=1}^N \left(\frac{aj}{N}\right)^2 \left(\frac{a}{N}\right) = \left(\frac{a}{N}\right)^3 \sum_{j=1}^N j^2 = \left(\frac{a}{N}\right)^3 \frac{N(N+1)(2N+1)}{6}.$$

We now compute the limit as $N \to \infty$:

$$\lim_{N\to\infty} a^3 \frac{N(N+1)(2N+1)}{6N^3} = \frac{a^3}{3}\,.$$

Thus

$$\int_0^a x^2\,dx = \frac{a^3}{3}\,.$$

B–2. Let a_n be a *bounded* sequence of real numbers. If $c > 1$, show that the series $\displaystyle\sum_1^\infty \frac{a_n}{n^x}$ converges uniformly for $x \geq c$.

Solution: We use the Weierstrass M-test. There is a constant K so that $|a_n| \leq K$. Since $x \geq c$ then $|\frac{a_n}{n^x}| \leq \frac{K}{n^c}$. Because $c > 1$, the series $\sum \frac{K}{n^c}$ converges. Thus, by the Weierstrass M-test, the series $\sum_1^\infty \frac{a_n}{n^x}$ converges uniformly and absolutely for $x \geq c$.

B–3. Let $f(x) \in C([0,1])$ be a continuous function with the property: $\displaystyle\int_0^1 f(x)p(x)dx = 0$ for *every* polynomial $p(x)$. Show that $f(x) \equiv 0$.

Solution: Since $f(x)$ is continuous on the compact set $[0,1]$, it is bounded, so $|f(x)| \leq M$ for some M. By the Weierstrass approximation theorem, there is a polynomial $p(x)$ so that in the uniform norm on $[0,1]$,

$$\|f - p\| < \epsilon.$$

Then

$$\int_0^1 f^2(x)\,dx = \int_0^1 f(x)[f(x) - p(x)]\,dx + \int_0^1 f(x)p(x)\,dx. \tag{1}$$

But

$$\left|\int_0^1 f(x)[f(x) - p(x)]\,dx\right| \leq M\epsilon, \tag{2}$$

while, by hypothesis, the second integral in (1) is zero. Thus

$$\int_0^1 f^2(x)\,dx \leq M\epsilon.$$

Since we can choose ϵ arbitrarily small, the only possibility is $f(x) \equiv 0$.

ALTERNATE: This is a replacement for (2). By the Schwarz inequality

$$\int_0^1 f(x)[f(x) - p(x)]\,dx \leq \sqrt{\int_0^1 f^2(x)\,dx}\sqrt{\int_0^1 [f(x) - p(x)]^2\,dx} \leq \epsilon\sqrt{\int_0^1 f^2(x)\,dx}.$$

Thus, from (1),

$$\sqrt{\int_0^1 f^2(x)\,dx} \leq \epsilon.$$

The advantage of this alternate proof is that it can be used for functions which are not necessarily continuous. The critical assumptions are that the integral $\int_0^1 f^2(x)\,dx$ exists and that for some polynomial we have $\sqrt{\int_0^1 [f(x)-p(x)]^2\,dx} \le \epsilon$.

B–4. Let

$$p(x) := (x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = x^6 - 21x^5 + \cdots$$

Clearly $p(4) = 0$. Denote by $p(x,t)$ the polynomial obtained by replacing $-21x^5$ by $-(21+t)x^5$, with $|t|$ small. Let $x(t)$ denote the perturbed value of root $x = 4$, so $x(0) = 4$.

a) Show that $x(t)$ is a smooth function of t for all $|t|$ sufficiently small.

Solution: We want to solve $p(x,t) = 0$ for $x = x(t)$ near $x = 4$, $t = 0$. But $p(4,0) = 0$ and

$$\frac{\partial p(x,0)}{\partial x}\bigg|_{x=4} = (x-1)(x-2)(x-3)(x-5)(x-6)\bigg|_{x=4} = 12.$$

Since this is not zero, by the implicit function theorem we can solve $p(x,t) = 0$ for $x = x(t)$ near $x = 4$, $t = 0$. Because $p(x,t)$ is a smooth function of x and t, so $x(t)$.

b) Compute the sensitivity of this root as one changes t, that is, compute $dx(t)/dt\big|_{t=0}$.

Solution: Take the derivative with respect to t of $p(x(t),t)$ =0 to find

$$0 = \frac{\partial p}{\partial x}\frac{dx}{dt}\bigg|_{\substack{x=4\\t=0}} + \frac{\partial p}{\partial t}\bigg|_{\substack{x=4\\t=0}} = 12\frac{dx(t)}{dt}\bigg|_{t=0} - x^5\bigg|_{x=4} = 12\frac{dx(t)}{dt}\bigg|_{t=0} - 1024..$$

Thus

$$\frac{dx(t)}{dt}\bigg|_{t=0} = \frac{1024}{12} \approx 85,$$

which is quite large.

B–5. Let $f(x)$ and $h(x,y)$ be continuous for x and y in the interval $[0,\,2]$. Show that if $\lambda > 0$ is sufficiently small, the equation

$$u(x) = f(x) + \lambda \int_0^2 h(x,y)u(y)\,dy \tag{3}$$

has a unique solution (that is, a solution exists and is unique).

Solution: Let B be the space $C([0,\,2])$ with the uniform norm, $\|u\| = \max_{0\le x\le 2}|u(x)|$. This is a complete metric space. Define the map

$$T(u)(x) = f(x) + \lambda \int_0^2 h(x,y)u(y)\,dy.$$

This map T clearly maps B to itself (why?). We'll show it is contracting for all sufficiently small λ. If $u(x)$, $v(x) \in C([0,\,2]$, then

$$T(u)(x) - T(v)(x) = \lambda \int_0^2 h(x,y)[u(y)-v(y)]\,dy.$$

Because $h(x, y)$ is continuous for x, y in the compact set $[0, 2]$, it is bounded there, so $|h(x, y)| \leq M$ for some M. Thus,

$$|T(u)(x) - T(v)(x)| \leq 2\lambda M \|u - v\|.$$

Because this holds for all $x \in [0,\, 2]$, we conclude that

$$\|T(u) - T(v)\| \leq 2\lambda M \|u - v\|.$$

Pick some λ with $|\lambda| < 1/(2M)$, we conclude that the map T is a contraction and hence has a unique fixed point $u \in B$: $u = T(u)$, that is, u is the unique solution of equation (3).