Signature	Printed Name

Math 509
April 28, 2005

Final Exam. Solutions

Jerry L. Kazdan
11:00 — 1:00

Directions This exam has 6 traditional problems (15 points each). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Score	
1	
2	
3	
4	
5	
6	
Total	

1. Let $\{a_n\}$ be a bounded sequence and $f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^x}$. Given any $c > 1$, show that this series converges uniformly in the interval $x \ge c$.

Solution: Say $|a_n| \le M$. If $x \ge c$ then $|a_n/n^x| \le M/n^c$. Since $c > 1$ the series $\sum M/n^c$ converges (integral test). Thus by the Weierstrass M Test the series $\sum a_n/n^x$ converges uniformly in the region $x \ge c$ for any $c > 1$.

2. Compute $\lim_{\lambda\to\infty} \int_0^{2\pi} |\cos \lambda x|\, dx$.

Solution: Let $t = \lambda x$. Then

$$J(\lambda) := \int_0^{2\pi} |\cos \lambda x|\, dx = \tfrac{1}{\lambda} \int_0^{2\pi\lambda} |\cos t|\, dt.$$

Now λ is between two integers, say $k \le \lambda < k+1$. Then

$$\int_0^{2\pi\lambda} = \int_0^{2\pi k} + \int_{2\pi k}^{2\pi\lambda} = I_1 + I_2.$$

But

$$I_1 = k \int_0^{2\pi} |\cos t|\, dt = 4k \quad \text{and} \quad 0 \le I_2 < 4.$$

Thus

$$J(\lambda) = \tfrac{1}{\lambda}[4k + I_2] \to 4.$$

3. a) If $f(x) \in C([-1,1])$ and

$$\int_{-1}^{1} f(x) x^n\, dx = 0 \qquad (1)$$

for all $n = 0, 1, 2, \ldots$, show that f must be identically zero.

Solution: Since f is continuous on the compact set $|x| \le 1$, it is bounded there: $|f(x)| \le M$. Given any $\epsilon > 0$, by the Weierstarss approximation theorem there is a polynomial $p(x)$ so that $|f(x) - p(x)| < \epsilon$ throughout the compact set $|x| \le 1$. The assumption implies that $\int_{-1}^{1} f(x)p(x)\, dx = 0$. Thus,

$$\int |f(x)|^2\, dx = \left| \int_{-1}^{1} f(x)[f(x) - p(x)]\, dx \right| \le 2M\epsilon.$$

Since ϵ can be chosen arbitrarily small, this implies that $\int |f(x)|^2\, dx = 0$ so $f(x) \equiv 0$.

b) If you only know that (1) holds for $n = 0, 2, 4, \ldots$ (so all *even* n), must f be identically zero? Explain.

Solution: Clearly for any ODD function $g(x)$ (such as $g(x) = x$) we have $\int_{-1}^{1} g(x)x^{2k}\,dx = 0$ for all integers k. Thus, the assumption does not imply that $f(x) \equiv 0$. However, if f is an even function, then (1) holds only for $f \equiv 0$.

The precise assertion is that if (1) holds for all even integers n, if and only if f is an odd function. To prove this, write $f = f_{\text{even}} + f_{\text{odd}}$, where $f_{\text{even}}(x) = \frac{1}{2}[f(x) + f(-x)]$ and $f_{\text{odd}}(x) = \frac{1}{2}[f(x) - f(-x)]$. Then

$$0 = \int_{-1}^{1} f(x)x^{2k}\,dx = \int_{-1}^{1} f_{\text{even}}(x)x^{2k}\,dx.$$

But by the Weierstrasse approximation theorem, we can approximate an even function uniformly by an even polynomial, so the proof in part a). applies.

4. The following equations define a map $F : (x, y, z) \mapsto (u, v, w)$:

$$\begin{aligned} u(x, y, z) &= x + xyz^2 \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + cx + 3z^2 \end{aligned}$$

Clearly $F : (1, 1, 0) \mapsto (1, 2, c)$. Write $p = (1, 1, 0)$ and $q = (1, 2, c)$.

a) Compute the derivative $F'(p)$.

Solution: $F'(x, y, z) = \begin{pmatrix} 1 + yz^2 & vxz^2 & 2xyz \\ y & 1 + x & 0 \\ c & 0 & 1 + 6z \end{pmatrix}$ so $F'(p) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ c & 0 & 1 \end{pmatrix}$.

b) For which value(s) of the constant c can the system of equations: can be solved for x, y, z as smooth functions of u, v, w near $(1, 1, 0)$? Justify your assertion(s).

Solution: Since $\det F'(p) = 2 \neq 0$ for any c, then by the inverse function theorem, for any value of c these equations can be solved for x, y, z as smooth functions of u, v, w near $(1, 1, 0)$

c) If c is one of these "good" values, let $G : (u, v, w) \mapsto (x, y, z)$ be the map inverse to F. Compute the derivative $G'(q)$ and use it to compute $\partial y(u, v, w)/\partial v$ at q.

Solution: By the inverse function theorem

$$G'(q) = [F'(p)]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ c & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -c & 0 & 1 \end{pmatrix}.$$

Consequently $\partial y(u, v, w)/\partial v = 1/2$ at q.

5. Let $f(x)$ and $K(x,y)$ be given continuous functions for $x, y \in [0,2]$. Consider the following linear *integral equation* for the continuous function $u(x)$:

$$u(x) = f(x) + \lambda \int_0^2 K(x,y)u(y)\,dy \tag{2}$$

If $|\lambda|$ is sufficiently small, show that this equation has a unique solution. [The choice of λ will depend on $M := \max_{x,y\in[0,2]} |K(x,y)|$.]

Solution: Use the contraction mapping theorem with the Banach space $C(-1,1])$ equipped with the uniform norm, and the map

$$Lu(x) := f(x) + \lambda \int_0^2 K(x,y)u(y)\,dy.$$

Clearly $L : C(-1,1]) \to C(-1,1])$. We need only pick λ so it is contracting. Now

$$|Lu(x) - Lv(x)| \le \left|\lambda \int_0^2 K(x,y)[u(y) - v(y)]\,dy\right| \le 2M|\lambda|\|u - v\|.$$

Since this holds for all $x \in [0,2]$ we have $\|Lu - Lv\| \le 2M|\lambda|\|u - v\|$, so L is contracting if $|\lambda| < 1/(2M)$.

6. a) Using the inner product $\langle f,\, g\rangle := \int_{-1}^{1} f(x)g(x)\,dx$ find an orthonormal basis for the space $\mathcal{S}$ spanned by the functions 1, x, and x^2.

Solution: We use the Gram-Schmidt process from linear algebra. Let $p_1(x) = 1/\|1\|$. Note $\|1\|^2 = \int_{-1}^{1} 1\,dx = 2$ so $p_1(x) = 1/\sqrt{2}$. Clearly x is orthogonal to 1 so the next element of the orthogonal basis is $p_2(x) = x/\|x\|$. Since $\|x\|^2 = \int_{-1}{1} x^2\,dx = 2/3$, we have $p_2(x) = \sqrt{\frac{3}{2}}\,x$. Although x^2 is orthogonal to x (and hence p_2), it is not orthogonal to 1. We thus let $q_3(x) = x^2 - \langle x^2,\, p_1\rangle p_1$ and then normalize to get $p_3(x) = q_3(x)/\|q_3\|$. The calculation is:

$$\langle x^2,\, p_1\rangle = \tfrac{1}{\sqrt{2}} \int_{-1}^{1} x^2\,dx = \frac{\sqrt{2}}{3}$$

so $q_3(x) = x^2 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$. Now $\|q_3\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2\,dx = \frac{8}{45}$. Thus $p_3(x) = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$.

b) Compute $\displaystyle \min_{a,b,c\in\mathbb{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx$.

Solution: This minimum, call it Q, is the square of the distance from x^3 to the subspace $\mathcal{S}$. We'll compute the orthogonal projection $\phi(x)$ of x^3 into $\mathcal{S}$ and then, by Pythagoras, $Q = \|x^3\|^2 - \|\phi\|^2$.

Now $\phi(x) = \langle x^3,\, p_1\rangle p_1 + \langle x^3,\, p_2\rangle p_2 + \langle x^3,\, p_3\rangle p_3$. But x^3 (by oddness) is orthogonal to p_1 and p_3. Also

$$\langle x^3,\, p_2\rangle = \sqrt{\tfrac{3}{2}} \int_{-1}^{1} x^3 x\,dx = \sqrt{\tfrac{3}{2}} \int_{-1}^{1} x^4\,dx = \sqrt{\tfrac{3}{2}}\,\tfrac{2}{5} = \tfrac{\sqrt{6}}{5}.$$

Consequently, $\|\phi\|^2 = \frac{6}{25}\|p_2\|^2 = \frac{6}{25}$. Since $\|x^3\|^2 = \int_{-1}^{1} x^6\,dx = \frac{2}{7}$ we see that

$$Q = \frac{2}{7} - \frac{6}{25} = \frac{8}{175}.$$

c) Compute $\max \int_{-1}^{1} x^3 h(x)\,dx$ where $h \in L^2(-1,1)$ is subject to the restrictions

$$\int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} xh(x)\,dx = \int_{-1}^{1} x^2 h(x)\,dx = 0; \qquad \int_{-1}^{1} |h(x)|^2\,dx = 1.$$

Solution:

IDEA: Let $\mathcal{S} \subset \mathbb{R}^n$ be a subspace and $Z \in \mathbb{R}^n$ a given vector. Then the unit vector $X \notin \mathcal{S}$ that is perpendicular to $\mathcal{S}$ with $J := \langle X, Z\rangle$ as large as possible is a unit vector in the direction of the projection of X orthogonal to $\mathcal{S}$.

With this in mind, let $\psi := x^3 - \phi$ be the projection of x^3 orthogonal to $\mathcal{S}$. Note that above we computed $Q = \|\psi\|^2$. We can write any function $f \in L^2(-1,1)$ as the orthogonal sum

$$f = f_{\mathcal{S}} + c\psi + g,$$

where $f_{\mathcal{S}}$ is the orthogonal projection of f into $\mathcal{S}$, c is a scalar, and g is orthogonal to 1, x, x^2, and x^3. Since h is orthogonal to $\mathcal{S}$, we can write $h = c\psi + g$. Then

$$\int_{-1}^{1} x^3 h(x)\,dx = \langle \phi + \psi,\, c\psi + g\rangle = c\|\psi\|^2 = cQ$$

so we want to make c as large as possible. But by Pythagoras, $1 = \|h\|^2 = c^2\|\psi\|^2 + \|g\|^2 \le c^2 Q$. Thus to make c largest, set $g = 0$ so $h = c\psi$ with $c = 1/\sqrt{Q}$. Then

$$\max \int_{-1}^{1} x^3 h(x)\,dx = \sqrt{Q} = \sqrt{\frac{8}{175}}.$$