Signature	PRINTED NAME

Math 509 — April 28, 2005

Final

Jerry L. Kazdan — 11:00 — 1:00

DIRECTIONS This exam has 6 traditional problems (15 points each). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Score	
1	
2	
3	
4	
5	
6	
Total	

1. Let $\{a_n\}$ be a bounded sequence and $f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^x}$. Given any $c > 1$, show that this series converges uniformly in the interval $x \ge c$.

2. Compute $\lim_{\lambda\to\infty} \int_0^{2\pi} |\cos \lambda x|\, dx$.

3. a) If $f(x) \in C([-1,1])$ and
$$\int_{-1}^{1} f(x)x^n\, dx = 0 \qquad (1)$$
for all $n = 0, 1, 2, \ldots$, show that f must be identically zero.

   b) If you only know that (1) holds for $n = 0, 2, 4, \ldots$ (so all *even* n), must f be identically zero? Explain.

4. The following equations define a map $F : (x, y, z) \mapsto (u, v, w)$:
$$u(x, y, z) = x + xyz^2$$
$$v(x, y, z) = y + xy$$
$$w(x, y, z) = z + cx + 3z^2$$
Clearly $F : (1, 1, 0) \mapsto (1, 2, c)$. Write $p = (1, 1, 0)$ and $q = (1, 2, c)$.

   a) Compute the derivative $F'(p)$.

   b) For which value(s) of the constant c can the system of equations: can be solved for x, y, z as smooth functions of u, v, w near $(1, 1, 0)$? Justify your assertion(s).

   c) If c is one of these "good" values, let $G : (u, v, w) \mapsto (x, y, z)$ be the map inverse to F. Compute the derivative $G'(q)$ and use it to compute $\partial y(u, v, w)/\partial v$ at q.

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5. Let $f(x)$ and $K(x,y)$ be given continuous functions for $x, y \in [0,2]$. Consider the following linear *integral equation* for the continuous function $u(x)$:

$$u(x) = f(x) + \lambda \int_0^2 K(x,y)u(y)\,dy \tag{2}$$

If $|\lambda|$ is sufficiently small, show that this equation has a unique solution. [The choice of λ will depend on $M := \max_{x,y\in[0,2]} |K(x,y)|$.]

6. a) Using the inner product $\langle f,\, g\rangle := \int_{-1}^1 f(x)g(x)\,dx$ find an orthonormal basis for the space $\mathcal{S}$ spanned by the functions 1, x, and x^2.

   b) Compute $\displaystyle \min_{a,b,c\in\mathbb{R}} \int_{-1}^1 |x^3 - a - bx - cx^2|^2\,dx$.

   c) Compute $\displaystyle \max \int_{-1}^1 x^3 h(x)\,dx$ where $h \in L^2(-1,1)$ is subject to the restrictions

$$\int_{-1}^1 h(x)\,dx = \int_{-1}^1 xh(x)\,dx = \int_{-1}^1 x^2 h(x)\,dx = 0; \qquad \int_{-1}^1 |h(x)|^2\,dx = 1.$$