Signature ______________________   Printed Name ______________________

Math 509
March 3, 2005

Exam 1

Jerry L. Kazdan
1:30 — 2:50

Directions This exam has two parts, Part A has 4 short answer problems (24 points) while Part B has 6 traditional problems (72 points). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Part A: Proof or Counterexample (4 problems, 6 points each)
Here let $f_n(x)$, $n = 1, 2, \ldots$ be a sequence of continuous functions for $0 \le x < \infty$ with $f_n(x) = 0$ for $x \ge n$. *For a counterexample, a clear sketch may be completely adequate.*

A–1. If $f_n(x)$ converges to zero for every $x \in [0,1]$, then f_n converges to zero *uniformly* on the interval $[0,1]$.

A–2. If $f_n(x)$ converges to zero for every $x \in [0,1]$, then $\int_0^1 f_n(x)\,dx \to 0$.

A–3. If $f_n(x)$ converges to zero uniformly for x in the interval $x \in [0,1]$, then $\int_0^1 f_n(x)\,dx \to 0$.

A–4. $f_n(x)$ converges to zero uniformly for x in the interval $0 \le x < \infty$, then $\int_0^\infty f_n(x)\,dx \to 0$.

Part B: Traditional Problems (6 problems, 12 points each)

B–1. Let $f \in C^2([0,3])$ have the properties $f(0) = 4$, $f(1) = 3$, and $f(3) = 6$. Show there is at least one point $z \in [0,3]$ where $f''(z) \ge$ const > 0 and give an estimate for this constant.

B–2. Let $f(x) \in C([0,2])$ be a continuous function with the property: $\int_0^2 f(x)h(x)dx = 0$ for *every* function $h \in C([0,2])$ that is zero at the end points: $h(0) = h(2) = 0$. Show that $f(x) \equiv 0$.

B–3. Let $f(x) \in C([0,1])$. Find $\lim_{n\to\infty} n \int_0^1 f(x)e^{-2nx}\,dx$ (justify your assertions).

[*continued on the next page*]

B–4. Let $\varphi_k(x)$, $x \in \mathbb{R}^2$, be a sequence of smooth functions with the following properties

i). $\varphi_k(x) \geq 0$ for $\|x\| < 1/k$, $\varphi_k(x) = 0$ for $\|x\| \geq 1/k$,

ii). $\iint_{\mathbb{R}^2} \varphi_k(x)\, dx = 1$.

For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set $\mathcal{K}$, define

$$f_k(x) := \iint_{\mathbb{R}^2} f(y)\varphi_k(x-y)\, dy.$$

a) Show that $\lim_{n\to\infty} f_k(x) = f(x)$, and that this convergence is uniform.

B–5. Compute $\displaystyle\iint_{R^2} \frac{dx\, dy}{[4+5x^2-2xy+2y^2]^{3/2}}$.

B–6. Let $x = (x_1, \ldots, x_n)$ and assume that $u(x)$ depends only on $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, so $u(x) = f(r)$ for some function f depending only on r.

a) Show that $\dfrac{\partial u}{\partial x_i} = \dfrac{df}{dr}\dfrac{x_i}{r}$.

b) Show that $\dfrac{\partial^2 u}{\partial x_i^2} = \dfrac{d^2 f}{dr^2}\left(\dfrac{x_i^2}{r^2}\right) + \dfrac{df}{dr}\left(\dfrac{1}{r} - \dfrac{x_i^2}{r^3}\right)$.

c) Compute $\Delta u := \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \ldots + \dfrac{\partial^2 u}{\partial x_n^2}$ in terms of f and its derivatives.

d) If $n = 3$, use this to find all functions $u(x) = f(r)$ that satisfy $\Delta u = 0$ for all $x \neq 0$.