Maximal Ideals in $C([0,1])$

For $c \in [0,1]$ let $I_c = \{f \in C([0,1])|. : f(c) = 0\}$. This is clearly an ideal.

Theorem I_c is a maximal ideal. Conversely, every maximal ideal in $C([0,1]$ (other than $C([0,1])$ itself) has this form.

Proof Say I_c is contained in some larger ideal J. We will show that $J = C([0,1])$ so then I_c is a maximal ideal.
Say $g \in J$ is not in I_c, so $g(c) \neq 0$. Then $h(x) := g(x)/g(c) \in J$ and $(h(x) - 1) \in I_c$. Consequently

$$1 = h(x) + [1 - h(x)] \in J.$$

Thus $J = C([0,1])$. Note this proof did not use the compactness of [0, 1].

Conversely, let J be any maximal ideal in $C([0,1])$. Assume J is not of the form I_c for any $c \in [0,1]$. Then for every $c \in [0,1]$ there ia an $f_c \in J$ with $f(c) \neq 0$. Since f_c is continuous, there is a neighborhood, V_c, of c where $f(x) \neq 0$. These open sets cover $[0,1]$.
Because $[0,1]$ is compact, there is a finite sub-cover: $[0,1] \subset V_{c_1} \cup V_{c_2} \cup \cdots \cup V_{c_N}$. Let

$$g(x) = f_{c_1}^2(x) + f_{c_2}^2(x) + \cdots + f_{c_N}^2(x).$$

Since the function $f_{c_j} \neq 0$ on V_{c_j} we have $g(x) > 0$ on $[0,1]$. Therefore $1/g(x) \in J$ and hence $1 = g(x)/g(x) \in J$. Consequently $J = C([0,1])$.