Problem Set 8

Due: Thurs. Nov. 6, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please read all of Chapter 6 in the Rudin text. Note that we will only discuss the Riemann integral, not the Riemann-Stieltjes integral.

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Use the definition of the derivative as the limit of a difference quotient to show that $\cos x$ is differentiable for all x. [You may use without proof that $\lim_{\theta\to 0}\sin\theta/\theta = 1$ and $\lim_{\theta\to 0}(1-\cos\theta)/\theta = 0$.]

2. Let $A(t)$ be an $n\times n$ matrix whose elements depend smoothly on $t\in\mathbb{R}$. Assume $A(t)$ is invertible at $t = t_0$.
   a) Compute the derivative of $A^2(t)$ in terms of A and A'.
   b) Show that $A(t)$ is invertible for all t near t_0. [Problem Set 5 #10].
   c) Show that $A^{-1}(t)$ is differentiable at $t = t_0$ and find a formula for it. Of course, from the special case of 1×1 matrices you have a guess what it should (roughly) be.
   d) Find a formula for the derivative of $A^{-2}(t)$ at $t = t_0$.

3. In class we proved that the only solution of the differential equation $u'(x) = u(x)$ with $u(0) = 1$ is $u(x) = e^x$.
   a) Use this to find the unique solution of $v' = v$ with $v(0) = c$, where c is a constant.
   b) Apply this to show that $e^{x+a} = e^a e^x$ for all real a and x.
   c) If for some constant γ the differentiable function $v(x)$ satisfies $v' - \gamma v \le 0$, show that $v(x) \le v(0)e^{\gamma x}$ for all $x \ge 0$. [Hint: Consider $g(x) := e^{-\gamma x}v(x)$.]

4. A continuous function is called *piecewise linear* if it consists only of straight line segments (see `https://en.wikipedia.org/wiki/Piecewise_linear_function`)

   Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that given any $\epsilon > 0$, there is a piecewise linear function $g : [a, b] \to \mathbb{R}$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in [a, b]$. In other words, any continuous function on $[a, b]$ can be approximated "uniformly" by a piecewise linear function.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.

   a) If $f'(1) = 0$, $f''(1) = 0$, $f'''(1) = 0$ and $f''''(1) > 0$, show that f has a local minimum at $x = 1$.

   b) If $f'(1) = 0$, $f''(1) = 0$, and $f'''(1) > 0$, what can you say about the behavior of f near $x = 1$?

6. Say a smooth function $u(x)$ is a solution of the differential equation

$$u'' + 3u' - (1 + x^2)u = 0.$$

   a) Show that u cannot have a positive local maximum (that is, a local maximum where u is positive).

   b) Similarly, show that u cannot have a negative local minimum.

   c) If $u(x)$ satisfies the above equation on the interval $[0, 2]$ with the boundary conditions $u(0) = 0$ and $u(2) = 0$, show that $u(x) = 0$ in $[0, 2]$.

   d) Generalize all of the above to solutions of

$$u'' + b(x)u' - c(x)u = 0 \quad \text{on} \quad \{\alpha \le x \le \beta\},$$

   where $b(x)$ and $c(x)$ are any continuous functions with $c(x) > 0$.

7. a) A strictly increasing, continuous, real-valued function f on an open interval $I \subset \mathbb{R}$ has an inverse function f^{-1} which is also strictly increasing, continuous, and defined on an open interval U. Suppose $f \in C^1(I)$ and $f'(t_0) > 0$ at some point $t_0 \in I$ [here $C^1(I)$ means the function is differentiable on I and this derivative is a continuous function).

   Prove that there is an open sub-interval $I' \subset I$ on which f^{-1} exists, is strictly increasing, and continuous.

   b) Using f^{-1} from the previous part, prove that $f^{-1} \in C^1(U')$ (where U' is its domain) and that

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)}$$

   if x is chosen to equal $f^{-1}(y)$. This is a special case of the *Inverse Function Theorem*, which you will most likely study further (in higher dimensions) in Math 509. [HINT: Let $a := f^{-1}(y)$ and $b := f^{-1}(y + h)$. What does the Mean Value Theorem say about $f(b) - f(a)$?]

8. Use the definition of the integral as a Riemann sum to compute $\int_0^b \sin x \, dx$. You will need the formula for $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta$; see

`http://www.math.upenn.edu/~kazdan/202F13/notes/sum-sin_kx.pdf`

9. Let $f(x) = \sin(1/x)$ for $0 < x \le 2/\pi$ while $f(0) = 3$. Show that f is Riemann integrable on the interval $[0,\ 2/\pi]$.

10. Let f be continuous on the interval $[a, b]$ and assume that $f(x) \ge 0$ for all $a \le x \le b$. Use the definition of the integral as a Riemann sum to show that if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ everywhere. [You will need to use that since f is continuous, if it is positive at some point, then it is positive in some interval containing the point.]

11. Prove the *Integral Intermediate Value Theorem*: If f is real and continuous on $[a, b]$, then there exists $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

Also, give an example showing that such a c may not exist if f is not continuous.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Say a function $u(x)$ satisfies the differential equation

$$u'' + b(x)u' + c(x)u = 0 \tag{1}$$

on the interval $[0,\ A]$ and that the coefficients $b(x)$ and $c(x)$ are both bounded, say $|b(x)| \le M$ and $|c(x)| \le M$ (if the coefficients are continuous, this is always true for some M).

a) Define $E(x) := \frac{1}{2}(u'^2 + u^2)$. Show that for some constant γ (depending on M) we have $E'(x) \le \gamma E(x)$. [SUGGESTION; use the inequality $2xy \le x^2 + y^2$.]

b) Use Problem 3(c) above to show that $E(x) \le e^{\gamma x}E(0)$ for all $x \in [0,\ A]$.

c) In particular, if $u(0) = 0$ and $u'(0) = 0$, show that $E(x) = 0$ and hence $u(x) = 0$ for all $x \in [0,\ A]$. In other words, if $u'' + b(x)u' + c(x)u = 0$ on the interval $[0,\ A]$ and that the functions $b(x)$ and $c(x)$ are both bounded, and if $u(0) = 0$ and $u'(0) - 0$, then the only possibility is that $u(x) \equiv 0$ for all $x \ge 0$.

d) Use this to prove the *uniqueness theorem*: if $v(x)$ and $w(x)$ both satisfy equation (1) and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(x) \equiv w(x)$ in the interval $[0,\ A]$.

[Last revised: November 7, 2014]