Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 6

DUE: Thurs. Oct. 23, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please re-read all of Chapter 4 and the first part of Chapter 5 (through page 108) of the Rudin text.

The following short True-False [T/F] questions are exercises that are *not* to be handed-in – but you should know how to solve them. For each, either provide a proof or give a counterexample.

T/F-1 There is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if and only if x is an integer.

T/F-2 If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = 0$ for all rational numbers x, then $f(x) = 0$ for all real x.

T/F-3 There exists some $x > 1$ such that $\frac{x^2+5}{3+x^7} = 1$.

T/F-4 The function $f(x) := |x|^3$ is continuous for all $x \in \mathbb{R}$.

T/F-5 Let f, g, and h be continuous on the interval $[0, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0, 2]$ such that $f(c) = g(c) = h(c)$.

T/F-6 a) If f is continuous on $\mathbb{R}$, then f is bounded.
b) If f is continuous on $[0, 1]$, then f is bounded.
c) If f is continuous on $\mathbb{R}$ and is bounded, then f attains its supremum.

THE FOLLOWING PROBLEMS SHOULD BE HANDED-IN.

1. Prove that $\cos x$ and $\sin x$ are continuous for all $x \in \mathbb{R}$. [You may use the usual formulas for $\cos(x + y)$ and $\sin(x + y)$.]

2. Let $f(x) := x^2 + 4x$. Clearly $\lim_{x\to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, find $\delta > 0$ so that $|x| < \delta$ implies that $|f(x)| < \epsilon$. Express δ as a function of ϵ. [You are not asked to find the *best* δ.]

3. Prove that there exists some $x \in [1, 2]$ such that $x^5 + 2x + 5 = x^4 + 10$.

4. Show that at any time there are at least two diametrically opposite points on the equator of the earth with the same temperature. Generalize.

5. Construct a function f with the property that there are sequences a_n and b_n converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded. Does there exist such a function f that is continuous at $x = 0$?

6. Let $f(a, n) := (1 + a)^n$, where a and n are positive.

   a) For constant a, how does $f(a, n)$ behave as $n \to \infty$? For constant n, how does $f(a, n)$ behave as $a \to 0$?

   b) Let $L \geq 1$ be a given real number. Prove that there exists a sequence $a_n \to 0$ and $f(a_n, n) \to L$ as $n \to \infty$. In other words, depending on the choice of a_n, the function f may approach any value.

7. Which of the following functions are uniformly continuous on $[0, \infty)$ – and why (or why not)?

   a). $f(x) = x \sin x$, b). $g(x) = e^x$, c). $h(x) = \frac{1}{1+x}$

8. Show that $f(x) := \sqrt{x}$ is continuous for all $x \geq 0$. Is it uniformly continuous there?

9. If (X, d_1) any (Y, d_2) are two metric spaces (the metrics are d_1 and d_2), these metric spaces are called *homeomorphic* if there is a continuous bijection $f : X \to Y$.

   a) Prove that $[0, 1]$ and $\mathbb{R}$ are *not* homeomorphic.

   b) Prove that $\mathbb{R}$ and $0 < x < \infty$ are homeomorphic.

   c) Prove that $\mathbb{R}^2$ and the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ are homeomorphic.

   d) Prove that $(-1, 1)$ and $\mathbb{R}$ are homeomorphic.

10. Let $f(x) := x \sin(1/x)$ for $x \neq 0$ while $f(0) := 0$.

    a) Prove that f is continuous for all real x.

    b) Is f uniformly continuous for $x \in [0, 2/\pi]$? Why?

    c) Is f uniformly continuous for all real x? Why?

11. Consider $\mathbb{R}^n$ with the Euclidean norm $|x|_2$ and let $\|x\|$ be any norm on $\mathbb{R}^n$.

    a) Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be the function $f(x) := \|x\|$. Show that f is continuous at every point of $\mathbb{R}^n$.

b) Show these norms are *equivalent* in the sense that there are constants $c_1 > 0$, $c_2 > 0$ such that for any $x \in \mathbb{R}^n$

$$c_1|x|_2 \le \|x\| \le c_2|x|_2.$$

[Suggestion: Look at the function $f(x) := \|x\|/|x|_2$ on the unit sphere $|x|_2 = 1$].

12. Let $f(x)$ be a continuous real-valued function with the property

$$f(x+y) = f(x) + f(y)$$

for all real x, y. Show that $f(x) = cx$ for some constant c.

13. [Partly from Rudin, p. 99 # 8]. Let $E \subset \mathbb{R}$ be a set and $f : E \to \mathbb{R}$ be uniformly continuous.

a) If E is a bounded set, show that $f(E)$ is a bounded set.

b) If E is not bounded, give an example showing that $f(E)$ might not be bounded.

14. If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on *all* of $\mathbb{R}$, show there are constants a and b so that

$$|f(x)| \le a + b|x|.$$

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 [Rudin, p. 98 # 3]. Let $\mathcal{M}$ be a metric space and $f : \mathcal{M} \to \mathbb{R}$ a continuous function. Denote by $Z(f)$ the *zero set of* f. These are the points $p \in \mathcal{M}$ where f is zero, $f(p) = 0$.

a) Show that $Z(f)$ is a closed set.

b) [See also Rudin, p. 101 #20] Given *any* set $E \in \mathcal{M}$, the distance of a point p to E is defined by

$$h(p) := \inf_{z \in E} d(p, z).$$

Show that h is a uniformly continuous function.

c) Use the previous part to show that given any *closed* set $E \in \mathcal{M}$, there is a continuous function that is zero on E and positive elsewhere.

B-2 [Rudin, p. 99 # 13 or #11, see also p. 98 #4] *extension by continuity* Let X be a metric space, $E \subset X$ a dense subset, and $f : E \to \mathbb{R}$ a uniformly continuous function. Show that f has a unique continuous extension to all of X. That is, there is a unique continuous function $g : X \to \mathbb{R}$ with the property that $g(p) = f(p)$ for all $p \in E$.

In your proof, show where it fails if you tried to apply your procedure to extend the function $f(x) := \sin(1/x)$ from $E := \{0 < x \le 1\}$ to all of $\{0 \le x \le 1\}$.

[REMARK: One generalize this by replacing $\mathbb{R}$ by any complete metric space.]

[Last revised: October 27, 2014]