Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 3

DUE: Thurs. Sept. 18, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please re-read all of Chapter 2 and the first part of Chapter 3 of the Rudin text.

REVISION NOTE: This revised version of Homework Set 3 has clarifications and hints for problems # 6, 10, and 11. Problem 7 is solved in the Class Notes on Compactness and Problem 12 is now a Bonus Problem.

1. Give an example of closed sets $V_j \subset \mathbb{R}$ so that $\cup_{j=1}^{\infty} V_j$ is open.

2. (Rudin, p.43 #5) Construct a bounded set of real numbers with exactly three limit points.

3. For the following subsets in an indicated metric space determine the interior and boundary points; describe the closure.
   a) $(0, 1] \subset \mathbb{R}$.
   b) $R^2 \subset \mathbb{R}^3$ (the coordinate plane $z = 0$).
   c) $\mathbb{Q} \subset \mathbb{R}$.
   d) The graph of the function $y = \begin{cases} \sin \frac{1}{x}, & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$.

4. Which of the following sets are compact – and why?
   a) $[0, 1] \subset \mathbb{R}$.
   b) $X = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \subset \mathbb{R}$.
   c) $[0, 1] \backslash \mathbb{Q} \subset \mathbb{R}$.

5. Define the set $S \in \mathbb{R}$ consisting of the points
$$x_k = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^k}$$
and let $c = \sup S$. Show that $c = 1$.

6. Let $\mathcal{M}_{k,n}$ denote the set of $k \times n$ real matrices. If $A, B \in \mathcal{M}_{k,n}$, let
$$\langle A, B \rangle = \operatorname{trace}(AB^t)$$
(here B^t is the transpose of B so AB^t is a square matrix – and its *trace* is the sum of the diagonal elements).

a) Show that this has all of the properties of an inner product (see Page 2 of `http://www.math.upenn.edu/~kazdan/508F14/Notes/Sep4-14.pdf`)

b) Define $|A|^2 = \langle A,\, A\rangle$ and the metric by $d(A,\, B) := |A - B|$. Show this has all the properties a metric.

c) If A and B are $n \times n$ matrices, show that $|AB| \le |A||B|$.
HINT: The ij element of AB is the inner product of the i^{th} row of A with the j^{th} column of B.
Alternate: Prove and use that for any i and j

$$\left[\sum_k a_{ik}b_{kj}\right]^2 \le \left[\sum_k a_{ik}^2\right]\left[\sum_k b_{kj}^2\right].$$

d) Use the above to show that $|A^\ell| \le |A|^\ell$ for any positive integer ℓ. Give an example where strict inequality can occur.

7. Let K be a compact subset of $\mathbb{R}^n$, and suppose that $x \in \mathbb{R}^n$ does not lie in K. Let $d = \inf\{d(x,y) \mid y \in K\}$, where $d(x,y)$ is the distance from x to y. Prove that there is a point $z \in K$ such that $d(x,z) = d$.

8. Let (X,d) be a metric space, and let $A, B \subset X$ be two subsets. Define the distance between A and B to be

$$\operatorname{dist}(A,B) := \inf_{\substack{x\in A\\ y\in B}} d(x,y). \tag{1}$$

Give an example (with proof) of a metric space (X,d) and two closed disjoint non-empty subsets A, B in X such that $\operatorname{dist}(A,B) = 0$.

9. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in the euclidean plane $\mathbb{R}^2$ that contains every point with rational coordinates. Let $r_n > 0$ be a sequence of positive real numbers such that $\sum_n r_n = 1$, and consider the set

$$U = \cup_{n\in\mathbb{N}} D(x_n, r_n),$$

where $D(x,r) = \{y \in \mathbb{R}^2 :\ |y-x| < r\}$ is the open disc centered at x with radius r.

a) Prove that U is an open and dense subset of $\mathbb{R}^2$.

b) Prove that if $L \subset \mathbb{R}^2$ is any straight line, then L cannot be contained in U.

10. PRODUCT TOPOLOGY. If A and B are two sets, then the *product set* $A \times B$ is the set of all pairs of points $(a,\, b)$ where $a \in A$ and $b \in B$. For instance $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ and $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Define a distance on $E_1 \times E_2$ by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

a) (Warmup) View $\mathbb{R}^2$ as $\mathbb{R} \times \mathbb{R}$. What is the distance from the point $(1, 2)$ to the origin $(0, 0)$? Sketch the "disk" centered at the origin with radius r.

b) More gnerally, show that $U \subset E_1 \times E_2$ is open if and only if for each point $(u_1, u_2) \in U$, there exist U_1 and U_2 open in E_1 and E_2 respectively, with $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \subset U$. [HINT: First do this for $\mathbb{R} \times \mathbb{R}$.]

11. This problem introduces the *p-adic topology* on the rational numbers $\mathbb{Q}$. It is used in number theory. In thinking about this question, think of a particular p, say the 3-adic topology.

Let p be a fixed prime number such as 3 (prime numbers are assumed to be positive). Given a non-zero rational number r, we can write it uniquely in the form

$$r = \frac{p^\nu n}{k} \tag{2}$$

where n and ν are integers, k is a positive integer, and neither n nor k is divisible by p. Define $\nu(r)$ to be the integer ν occurring in this expression [note $\nu(r)$ means ν is a function of r – *not* multiplication]. For rational $r \neq 0$ we define the norm

$$|r|_p = p^{-\nu} \quad \text{while} \quad |0|_p = 0.$$

EXAMPLE: If $x = 63/550 = 2^{-1} \cdot 3^2 \cdot 5^{-2} \cdot 7 \cdot 11^{-1}$, then

$|x|_2 = 2$, $|x|_3 = 1/9$, $|x|_5 = 25$, $|x|_7 = 1/7$, $|x|_{11} = 11$, and $|x|_{13} = 1$.

For $x, y \in \mathbb{Q}$, define the coresponding metric

$$d_p(x, y) = |x - y|_p.$$

Warmup: Compute $d_3(9, 0)$, $d_3(18, 0)$, $d_3(4, 15)$, $d_3(15, 2/9)$, and $d_3(15, 9/2)$.

a) Show that $(\mathbb{Q}, d_p)$ is a metric space, and that in fact $d(x, z) \leq \max(d(x, y), d(y, z))$.

b) Show that if $x \in N_r(a)$, then $N_r(x) = N_r(a)$, so that any point of the neighborhood $N_r(a)$ is a "center" of that neighborhood.

c) Show that given two neighborhoods $N_{r_1}(a_1)$ and $N_{r_2}(a_2)$, either they are disjoint or one is contained in the other.

Strange metric, isn't it?

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Define two real numbers x and y to be equal if $|x - y|$ is an integer, thus we have a "topological circle" whose circumference is one.

Let α be an irrational real number, $0 < \alpha < 1$ and consider its integer multiples, α, 2α, 3α .... Show that this set is dense in $0 \leq x \leq 1$.

[Last revised: September 16, 2014]