**Math 508, Fall 2014** **Jerry L. Kazdan**

# Problem Set 2

DUE: Thurs. Sept. 11, 2014. *Late papers will be accepted until 1:00 PM Friday.*

**This week**. Please re-read all of Chapter 1 and read all of Chapter 2 of the Rudin text.

1. [From Problem Set 0]
   a) Let $a > 0$ be a rational number with the property that $a^2 < 2$. Find a rational number $b > a$ with the property $a^2 < b^2 < 2$, so $b$ is closer to $\sqrt{2}$.
   b) Similarly, let $c > 0$ be a rational number with the property that $2 < c^2$. Find a rational number $0 < d < c$ with the property $2 < d^2 < c^2$.
   c) The previous parts concerned solving $x^2 = 2$. If $c \in \mathbb{R}$ with $1 < c$, generalize part a) to solving $x^3 = c$.

2. Let $F$ be an ordered field and $x,\, y \in F$.
   a) If $x < y$, show that $x < \dfrac{x+y}{2} < y$.
   b) For each $x \in F$, if $x \neq 0$, then $x^2 > 0$.
   c) If $x^2 + y^2 = 0$, then $x = y = 0$.
   d) Show that $2xy \le x^2 + y^2$. When can equality occur?

3. a) Let $x < y$ be rational numbers. Show there is an irrational number between them.
   b) Let $x < y$ be real numbers. Show there is a rational number between them.

4. Let $f(x,y) := (x^2+y^2)^2 + y^4 + 2xy - x + 7y$. Note that $f(0,0) = 0$.
   Find some explicit number $R$ so that if $x^2 + y^2 \ge R^2$, then $f(x,y) \ge 1$.
   MORAL: the global minimum of $f$, if one exists, is inside the disc $x^2 + y^2 \le R^2$. [You are not asked to find the *best* $R$].

5. a) Find all non-empty bounded sets $A \subset \mathbb{R}$ such that $\sup A \le \inf A$.
   b) Let $A$ and $B$ be sets of real numbers. If $A$ is bounded above and $B$ is bounded below, prove that $A \cap B$ is bounded.

6. Let $z, w, v \in \mathbb{C}$ be complex numbers.
   a) Show that $|z - w| \ge |z - v| - |v - w|$.
   b) Graph the points $z = x + iy$ in the complex plane that satisfy $1 < |z - i| < 2$.

c) Let $z, w \in \mathbb{C}$ be complex numbers with $|z| < 1$ and $|w| = 1$. Show that

$$\left| \frac{w - z}{1 - \bar{z}w} \right| = 1.$$

7. a) Let $z, w, v \in \mathbb{C}$ and define $d(z, w)) := \dfrac{|z - w|}{1 + |z - w|}$. Show that

$$d(z, v) \leq d(z, w) + d(w, v) \qquad \text{[triangle inequality]}.$$

[Note that in this metric $d(z, w) < 1$ for all points $z$, $w$. Thus *all* sets are bounded. This metric and the usual metric (absolute value) have the same convergent sequences.]

b) Let $S$ be an arbitrary set with $p, q, r \in S$. Say there is a function $g : S \times S \to \mathbb{R}$ that satisfies the triangle inequality

$$g(p, r) \leq g(p, q) + g(q, r).$$

Define $d(p, q)) := \dfrac{g(p, q)}{1 + g(p, q)}$. Show that this function $d(p, q)$ also satisfies the triangle inequality.

8. The point of this problem is to give an example of an ordered field that does *not* have the Archimedean property.

Consider the set $\mathcal{R}$ of rational functions $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and $q(x)$ not identically zero. The function $f(x)$ has a finite value everywhere except at a finite numnber of points (the zeroes of $q(x)$).

a) It should be obvious that with the usual definitions of addition and multiplication, the set of rational functions is closed under addition and multiplication, that is, the sum and product of two rational functions ia also a rational function. Show that $\mathcal{R}$ forms a field.

b) In $\mathcal{R}$, define the order $f > 0$ to mean that $f(x) > 0$ for all sufficiently large positive real $x$. Thus, if

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_k x^k}{b_0 + b_1 x + \cdots + b_n x^n}$$

with $a_k \neq 0$ and $b_n \neq 0$, then $f > 0$ means $\dfrac{a_k}{b_n} > 0$. [This gives an algebraic definition of $f > 0$ that avoids defining "sufficiently large".] Then $f > g$ is defined to mean $f - g > 0$. Show that with this order relation, $\mathcal{R}$ is an ordered field.

c) Show that this ordered field is *non-archemedean* by exhibiting two specific rational functions $f$ and $g$ with the property that there is *no* integer $N$ such that $Nf > g$.

## Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Prove that if $X$ and $Y$ are two sets such that there are injective maps $f : X \to Y$ and $g : Y \to X$, then there is a bijection from $X$ to $Y$.

[Last revised: June 22, 2015]